Math 211-003: Review For Exam 1

Exam date 10/02/2008 in class

Be familiar with the terms **normal form** and **explicit/implicit/general/particular solutions**.

Separable Equations

An equation is called **separable** if it can be expressed as

$$\frac{dy}{dt} = f(t)g(y)$$

To solve such equations, divide by $g(y)$ and integrate

$$\int \frac{dy}{g(y)} = \int f(t)dt$$

Note that we have to assume that $g(y) \neq 0$ to solve in this way, so we need to see if $g(y) = 0$ can lead us to any more solutions.

For example, the ODE $x' = x$ has two solutions:

$$x(t) = Ce^t \text{ for } C \neq 0 \text{ and } x(t) = 0$$

The first solution family is obtained by solving and integrating; the second solution comes from checking the case $x = 0$. Of course we can combine the two answers to get one clean answer to this ODE, namely $x(t) = Ce^t$ for all real C.

See 2.2)#1,3,5,7,9,11,19,21,37,40 for practice on this type of problem

Linear Equations

An ODE is called **linear** if it may be expressed as

$$\frac{dy}{dt} = a(t)y + f(t)$$

If $f(t) = 0$ then we call the equation **homogeneous** and the resulting equation is **separable** with solution set $y(t) = Ce^{\int a(t)dt}, \forall t \in \mathbb{R}$. So we will look for solutions to **inhomogeneous** linear ODEs...

Our first method to solve such problem is the **integrating factor method**. Here, we find a function $\mu(t) = e^{-\int a(t)dt}$ such that

$$(\mu y)' = \mu y' - \mu a(t)y \Rightarrow (\mu y)' = \mu(y' - a(t)y) = \mu f(t)$$

$$\Rightarrow \mu(t)y(t) = \int \mu(t)f(t)dt + C \Rightarrow y(t) = \frac{1}{\mu(t)}\left(\int \mu(t)f(t)dt + C\right)$$

An alternative method is known as the **variation of parameters method**. First, we solve the **homogeneous part** of the ODE (just $y' = a(t)y$) and find the solution to this (called the **homogeneous solution**) which we denote y_h.

Once we find y_h we assume that our solution is of the form $y = vy_h$ where $v(t)$ is a **variable parameter**. Putting this back into the original equation yields

$$v(t) = \int \frac{f(t)}{y_h(t)} dt + C \Rightarrow y(t) = y_h(t)(\int \frac{f(t)}{y_h(t)} dt + C)$$

Both methods will yield the same result (look at the relationship between y_h and μ to confirm this).

See 2.4)#1,3,5,7,12,31,33,35,37,39 for practice on this type of problem

Mixing Problems

These problems are set up where a function $x(t)$ describes the amount of a particular solute in a solvent. The volume of the tank $v(t)$ can vary with time or remain constant. To set up the IVP, we need to find X_{in} and X_{out}, which describe the rate in/out of the solute into the tank. So we get,

$$\frac{dx}{dt} = X_{in} - X_{out}, x(0) = x_0$$

Where x_0 is the initial amount of the solute in the solution. Normally, X_{in} is a function only of t or a constant, and X_{out} is a function of x (and t).

If more than one tank is involved in the problem, we end up making a **system of equations**

$$\frac{dx}{dt} = X_{in} - X_{out}, x(0) = x_0$$

$$\frac{dy}{dt} = Y_{in} - Y_{out}, y(0) = y_0$$

where x,y describe the amount of the solute in each tank. In this case, one of the equations will probably contain both x and y. To solve these with our current toolbox, solve the ODE that only concerns one function, then use that result to solve for the other.

See 2.5)#3,5,7,12 for practice on this type of problem

Exact Equations

A **differential form** $\omega = P(x,y)dx + Q(x,y)dy$ is called **exact** if $\omega = dF$ for some differentiable function $F(x,y)$. We have the following theorem

$$\omega \text{ is } \textbf{exact} \iff \frac{\partial}{\partial y}P = \frac{\partial}{\partial x}Q$$

So if we know that an ODE of the form $\omega = Pdx + Qdy = 0$ is exact, then for F s.t. $\omega = dF$, $F(x,y) = C$ is a set of solutions to the ODE. We use the following to solve for F:

$$\frac{\partial}{\partial x}F = P \text{ and } \frac{\partial}{\partial y}F = Q$$

$$\Rightarrow F(x,y) = \int Pdx + \phi(y)$$

$$\Rightarrow \frac{\partial}{\partial y}F = \frac{\partial}{\partial y}\int Pdx + \phi'(y) = Q$$

$$\Rightarrow \phi'(y) = Q(x,y) - \frac{\partial}{\partial y}\int Pdx$$

Solving for ϕ will give us the answer.

Not every differential form is exact, so we are concerned with finding an **integration factor** μ such that $\mu\omega = \mu Pdx + \mu Qdy = 0$ is exact, since the solution to this new ODE will be a solution to $\omega = 0$ as well.

A diff. form ω is called **separable** if it is of the form

$$\omega = A(x)B(y)dx + C(x)D(y)dy$$

In this case the integrating factor is $\mu = \frac{1}{B(y)C(x)}$. So we have

$$\mu\omega = \frac{A(x)}{C(x)}dx + \frac{D(y)}{B(y)}dy$$

So then we can integrate to conclude that

$$F(x,y) = \int \frac{A(x)}{C(x)}dx + \int \frac{D(y)}{B(y)}dy = constant$$

will be the solution set to $\omega = 0$.

Another type of diff. form is **homogeneous***. In this case $\omega = Pdx + Qdy$ satisfies

$$P(tx,ty) = t^n P(x,y), Q(tx,ty) = t^n Q(x,y)$$

For a certain n and for all $t > 0$. For $\omega = 0$ when ω is homogenous, we try the substitution $y = vx$ ($\Rightarrow dy = vdx + xdv$) to get

$$P(x, vx)dx+Q(x, vx)(vdx+xdv) = 0 \Rightarrow x^n(P(1, v)+vQ(1, v))dx+x^{n+1}Q(1, v)dv = 0$$

which (while messy) is separable and can be solved. Once the solution is found in terms of x and v, replace v with $\frac{y}{x}$ and simplify.

To find the **slope** of a family of curves $F(x, y) = C$, solve for $\frac{dy}{dx}$ in $dF = 0$.

See 2.6)#9,11,13,15,17,19,21,23,25,30,35,37,39,42,43 for practice

Existence & Uniqueness of Solutions

For an IVP in normal form,

$$\frac{dy}{dt} = f(t, y), y(t_0) = y_0$$

The **Existence Theorem** states that, if f is continuous on a rectangle R containing (t_0, y_0), then a solution is guaranteed to exist at least until the curve $(t, y(t))$ leaves R. Remember that the width (the length in the t direction) of R does not necessarily mean that the solution $y(t)$ is defined for that same domain.

If we can also say that $\frac{\partial}{\partial y}f$ is continuous on R, then we know that the solution is **unique** (thanks to, not surprisingly, the **Uniqueness Theorem**).

The **Uniqueness Theorem** tells us that for f satisfying the conditions of the theorem, no two solutions can cross on rectangle R. So given one solution y_1 to $y' = f(t, y)$, we know that as long as the theorem applies, other solutions that are above y_1 at one point remain above y_1 for all applicable t (likewise if another solution is below y_1, we get a similar result).

See 2.7)#1,3,5,27,29,31 for practice