

Math 211-003: Review For Exam 2

Second Order Equations

A general **Second Order Linear ODE** is of the form:

$$y'' + p(t)y' + q(t)y = f(t)$$

For IVP's, there are **two** conditions, $y(t_0) = y_0$ and $y'(t_0) = y_1$. If $f = 0$, we call the ODE **homogeneous** and the general solution has the form:

$$y(t) = C_1y_1(t) + C_2y_2(t)$$

Where y_1 and y_2 are **linearly independent** solutions to the ODE. Otherwise, the general solution is of the form

$$y(t) = C_1y_1(t) + C_2y_2(t) + y_p(t)$$

Where y_p is a particular solution to the ODE.

To check for **independence**, we check the **Wronskian** of two functions.

$$W(u, v) = \det \begin{pmatrix} u & v \\ u' & v' \end{pmatrix} = uv' - vu'$$

For two solutions to an ODE, either $W(y_1, y_2) \neq 0$ or $W(y_1, y_2) = 0$ over the interval that the ODE is valid. In the first case, the solutions are independent. A **fundamental set** of solutions is a pair of independent solutions.

When we can't easily solve for solutions, we may be able to find second independent solution **assuming that y_1 is given**. In this case, we try a solution of the form $y_2 = vy_1$, where y_1 is given and v is an unknown function.

Constant Coefficients

When p and q are constants and $f = 0$, we try solutions of the form $y(t) = e^{\lambda t}$. This leads us to look for solutions to the **characteristic polynomial**:

$$\lambda^2 + p\lambda + q = 0$$

So our choice of solutions depend on whether our roots are

- **distinct real roots:** $y_1(t) = e^{\lambda_1 t}$ and $y_2(t) = e^{\lambda_2 t}$.
- **one repeated real root:** $y_1(t) = e^{\lambda t}$ and $y_2(t) = te^{\lambda t}$.
- **complex conjugate roots:** For $\lambda = \alpha \pm i\beta$, $y_1(t) = e^{\alpha t} \cos \beta t$ and $y_2(t) = e^{\alpha t} \sin \beta t$.

When $f \neq 0$, we use the **Method of Undetermined Coefficients**. We look for terms in f that we can try for y_p . For example, if $f = \sin t$, we try $y_p = A \cos t + B \sin t$. We need to make sure that no part of y_p exists as part of the set of homogeneous solutions. When $y_1 = \cos t$ and $y_2 = \sin t$, then (in our previous example) we will try $y_p = At \cos t + Bt \sin t$. The essential rule is "when we get in trouble, multiply the trial solution by t ."

The Laplace Transform

The for a piece-wise continuous function f on $[0, \infty)$, the **Laplace of f** is

$$F(s) = \mathcal{L}\{f(t)\}(s) = \int_0^{\infty} f(t)e^{-st} dt$$

Some properties of the Laplace are:

- $\mathcal{L}\{\alpha f + \beta g\}(s) = \alpha F(s) + \beta G(s)$
- $\mathcal{L}\{e^{ct}f(t)\}(s) = F(s - c)$
- $\mathcal{L}\{t^n f(t)\}(s) = (-1)^n F^{(n)}(s)$
- $\mathcal{L}\{y^{(n)}(t)\}(s) = s^n Y(s) - \sum_{k=0}^{n-1} s^{n-1-k} y^{(k)}(0)$

The (almost) **inverse** of the Laplace is found indirectly. When a function $F(s)$ is **rational**, it may be expressed as a sum of **partial fractions**. For example, for $F(s) = \frac{s+1}{s^2-4s+3} = \frac{2}{s-3} - \frac{1}{s-1}$. So $\mathcal{L}^{-1}\{F\}(t) = 2e^{3t} - e^t$.

The **Heaviside Function** is

$$H(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

We can use this to work with discontinuous functions. For example if f is

$$f(t) = \begin{cases} t & 0 \leq t < 4 \\ 0 & 4 \leq t < 5 \\ 2 & t \geq 5 \end{cases}$$

then it may be expressed as $f(t) = t + H(t-4)(-t) + 2H(t-5)$.

We have the following rules for the Heaviside function:

- $\mathcal{L}\{H(t-c)f(t-c)\}(s) = e^{-cs}F(s)$
- $\mathcal{L}^{-1}\{e^{-cs}F(s)\}(t) = H(t-c)f(t-c)$

So in the previous example, $f(t) = f_0(t) + H(t-4)f_1(t-4) + H(t-5)f_2(t-5)$, where $f_0(t) = t$, $f_1(t) = -t - 4$, and $f_2(t) = 2$. So

$$\mathcal{L}\{f\}(s) = \frac{1}{s^2} - e^{-4s} \left(\frac{1}{s^2} + \frac{4}{s} \right) + \frac{2e^{-5s}}{s}$$

The **convolution** of f and g is

$$(f * g)(t) = \int_0^t f(u)g(t-u) du$$

We have that $\mathcal{L}^{-1}\{F(s)G(s)\}(t) = (f * g)(t)$. Specifically, we have that

$$\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\}(t) = (1 * f)(t) = (f * 1)(t) = \int_0^t f(u) du$$

Using the Laplace Transform

Our strategy is to use the Laplace on an IVP to solve an algebraic equation. Once a solution is found, we use the inverse Laplace to get a solution to our initial IVP.

EX) Suppose $y(t)$ satisfies $y(0) = 0$, $y'(0) = 1$, and $y'' - 2y' + y = e^t + 1$. Then we get (through the Laplace)

$$s^2Y(s) - 1 - 2sY(s) + Y(s) = \frac{1}{s-1} + \frac{1}{s}$$

or

$$\begin{aligned} Y(s) &= \frac{1}{(s-1)^3} + \frac{1}{s(s-1)^2} + \frac{1}{(s-1)^2} \\ &= \frac{1}{2} \frac{2}{(s-1)^3} + 2 \frac{1}{(s-1)^2} - \frac{1}{s-1} + \frac{1}{s} \end{aligned}$$

So

$$y(t) = \mathcal{L}^{-1}\{Y\}(t) = \frac{t^2 e^t}{2} + 2te^t - e^t + 1$$

Practice Problems

- Sec 4.1) #17,19,23,25,27,29
- Sec 4.3) odds
- Sec 4.5) #19,21,23,25,27,29,33,37,41,43
- Sec 5.3) #11,13,19,21,23,25,27,35
- Sec 5.4) #11,13,15,17,21,27,29,31
- Sec 5.5) #1,5,7,11,13,15,19,25
- Sec 5.7) #5,7,9,11,13,15,17,19,21,23
- Questions on HW4-HW7 not included in the book

If you're more comfortable with one method of solving IVP's over another, just go nuts and solve any IVP your preferred way. For instance, if you think Laplace is where it's at, solve the questions from Chapter 4 using Laplace instead.