

# Inverse Laplace

For People Who Don't Like Partial Fractions

Let's look at an alternative method to finding the inverse Laplace in situations where taking partial fractions can become tedious and open us up to many potential mistakes:

## The Method, Part 1

#EX1) Find

$$\mathcal{L}^{-1} \left\{ \frac{1}{s^4(s-2)} \right\} (t)$$

If we solve this using partial fractions, it is entirely possible. However, we end up solving this equation:

$$\frac{1}{s^4(s-2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s^4} + \frac{E}{s-2}$$

Let us instead consider the following relationship:

$$\mathcal{L}^{-1} \left\{ \frac{F(s)}{s} \right\} (t) = \int_0^t f(u) du$$

We know that

$$\mathcal{L}^{-1} \left\{ \frac{1}{s-2} \right\} (t) = e^{2t}$$

So

$$\begin{aligned} \mathcal{L}^{-1} \left\{ \frac{1}{s(s-2)} \right\} (t) &= \mathcal{L}^{-1} \left\{ \frac{1}{s} \cdot \frac{1}{s-2} \right\} (t) = \int_0^t e^{2u} du \\ &= \left[ \frac{e^{2u}}{2} \right]_0^t = \frac{e^{2t}}{2} - \frac{1}{2} \end{aligned}$$

Again we can use this same relationship

$$\begin{aligned} \mathcal{L}^{-1} \left\{ \frac{1}{s^2(s-2)} \right\} (t) &= \mathcal{L}^{-1} \left\{ \frac{1}{s} \cdot \frac{1}{s(s-2)} \right\} (t) = \int_0^t \frac{e^{2u}}{2} - \frac{1}{2} du \\ &= \left[ \frac{e^{2u}}{4} - \frac{u}{2} \right]_0^t = \frac{e^{2t}}{4} - \frac{t}{2} - \frac{1}{4} \end{aligned}$$

Again

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s^3(s-2)}\right\}(t) &= \int_0^t \frac{e^{2u}}{4} - \frac{u}{2} - \frac{1}{4} du \\ &= \left[\frac{e^{2u}}{8} - \frac{u^2}{4} - \frac{u}{4}\right]_0^t = \frac{e^{2t}}{8} - \frac{t^2}{4} - \frac{t}{4} - \frac{1}{8}\end{aligned}$$

And finally we get that

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s^4(s-2)}\right\}(t) &= \int_0^t \frac{e^{2u}}{8} - \frac{u^2}{4} - \frac{u}{4} - \frac{1}{8} du \\ \left[\frac{e^{2u}}{16} - \frac{u^3}{12} - \frac{u^2}{8} - \frac{u}{8}\right]_0^t &= \frac{e^{2t}}{16} - \frac{t^3}{12} - \frac{t^2}{8} - \frac{t}{8} - \frac{1}{16}\end{aligned}$$

This solution entirely agrees with the partial fraction decomposition:

$$\frac{1}{s^4(s-2)} = -\frac{1}{16} \cdot \frac{1}{s} - \frac{1}{8} \cdot \frac{1}{s^2} - \frac{1}{4} \cdot \frac{1}{s^3} - \frac{1}{2} \cdot \frac{1}{s^4} + \frac{1}{16} \cdot \frac{1}{s-2}$$

So we see the trade-off: either a lot of algebra with partial fractions or 4 (relatively) basic integrations.

## The Method, Part 2

The previous method works well with taking the inverse Laplace of fractions of the form:

$$\frac{1}{s^n(s-a)}, \text{ for integers } a, n$$

But what if the piece under the exponent is not just  $s$ ?

#EX2) Find

$$\mathcal{L}^{-1} \left\{ \frac{1}{(s-1)^2(s+1)} \right\} (t)$$

For these questions, we need one more relationship:

$$\mathcal{L} \{ e^{at} f(t) \} (s) = F(s-a) \text{ or } \mathcal{L}^{-1} \{ F(s-a) \} (t) = e^{at} f(t)$$

So for the above problem:

$$F(s) = \frac{1}{(s-1)^2(s+1)} \text{ and } \mathcal{L}^{-1} \{ F(s) \} (t) = f(t)$$

We want to make  $F(s)$  have the same form as in part 1. So let's call  $G(s) = F(s+1) = F(s - (-1))$ . Now we have

$$G(s) = \frac{1}{s^2(s+2)} \text{ and } g(t) = e^{-t} f(t)$$

We can solve this using our method. After 2 integrations, we get

$$g(t) = \frac{e^{-2t}}{4} + \frac{t}{2} - \frac{1}{4}$$

Now we just use our new relationship to find what  $f(t)$  is:

$$\begin{aligned} g(t) &= e^{-t} f(t) \\ \Rightarrow f(t) &= e^t \left( \frac{e^{-2t}}{4} + \frac{t}{2} - \frac{1}{4} \right) = \frac{e^{-t}}{4} + \frac{te^t}{2} - \frac{e^t}{4} \end{aligned}$$

So to solve more general problems, **shift** the function to make  $s$  appear under the exponent in the denominator, solve for the new function, and then get answer by **multiplying** by the appropriate exponent to "shift" back.