Math 212: Answers to Assignment 2

Section 2.2

#1) To show that $A = \{(x,y) | -1 < x, y < 1\}$ is open, we need to show that for any point $\vec{x_0} = (x_0, y_0)$, we can find an $\varepsilon > 0$ such that $D_\varepsilon(\vec{x_0}) \subset A$. Let $\varepsilon = min\{1 - |x_0|, 1 - |y_0|\}$. For any $\vec{x} = (x,y) \in D_\varepsilon(\vec{x_0})$ we have

$$\left\|\vec{x_0} - \vec{x}\right\| < \varepsilon \Rightarrow (x - x_0)^2 + (y - y_0)^2 < \varepsilon^2 \Rightarrow |x - x_0|, |y - y_0| < \varepsilon$$

So,

$$|x_0| \leq |x_0 - x| + |x| < 1 - |x_0| + 1 \Rightarrow 2|x_0| < 2 \Rightarrow |x_0| < 1$$

$$|y_0| \leq |y_0 - y| + |y| < 1 - |y_0| + 1 \Rightarrow 2|y_0| < 2 \Rightarrow |y_0| < 1$$

So, $D_\varepsilon(\vec{x_0}) \subset A \Rightarrow A$ is open. □

#5a) Since x^3y is continuous, the limit is equal the the function evaluated at $(0,1)$, so

$$\lim_{(x,y)\to(0,1)} x^3y = 0^31 = 0$$

#6a) As in #5a above, we just need to calculate $e^01 = 1$

#6b) This time,

$$f(x) = \frac{\sin^2 x}{x}$$

doesn't have a value at $x = 0$, but the limit on the top and the bottom of the fraction both equal to 0. So we will use L'Hopital's rule:

$$\lim_{x\to 0} \frac{\sin^2 x}{x} = \lim_{x\to 0} \frac{2\sin x \cos x}{1} = 0$$

#6c) Again, we need L'Hopital's rule (twice):

$$\lim_{x\to 0} \frac{\sin^2 x}{x^2} = \lim_{x\to 0} \frac{2\sin x \cos x}{2x} = \lim_{x\to 0} \frac{2\cos^2 x - 2\sin^2 x}{2} = 1$$

Section 2.3

#1a)

$$\frac{\partial}{\partial x} f = y$$
$$\frac{\partial}{\partial y} f = x$$

#1b) Using #1a and the chain rule:

$$\frac{\partial}{\partial x} f = ye^{xy}$$
$$\frac{\partial}{\partial y} f = xe^{xy}$$

#1c) We use the product rule for the first one:

$$\frac{\partial}{\partial x} f = \cos x \cos y + x(-\sin x \cos y) = (\cos x - x \sin x) \cos y$$
$$\frac{\partial}{\partial y} f = x \cos x(-\sin y) = -x \cos x \sin y$$

#3a) The product rule and chain rules are used here:

$$\frac{\partial}{\partial x} w = e^{x^2+y^2} + x(2x)e^{x^2+y^2} = (1 + 2x^2)e^{x^2+y^2}$$
$$\frac{\partial}{\partial y} w = x(2y)e^{x^2+y^2} = 2xye^{x^2+y^2}$$

#3d) We have to assume that $y \neq 0$. Using $w = xy^{-1}$, we use the power rule for the second:

$$\frac{\partial}{\partial x} w = \frac{1}{y}$$
$$\frac{\partial}{\partial y} w = -\frac{x}{y^2}$$

#7a)

$$f_1(x,y) = x \text{ and } f_2(x,y) = y$$

$$\frac{\partial}{\partial x} f_1 = 1$$

$$\frac{\partial}{\partial y} f_1 = 0$$

$$\frac{\partial}{\partial x} f_2 = 0$$

$$\frac{\partial}{\partial y} f_2 = 1$$

$$Df = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

#7b)

$$f_1(x,y) = xe^y + \cos y, f_2(x,y) = x, f_3(x,y) = x + e^y$$

$$\frac{\partial}{\partial x} f_1 = e^y \text{ and } \frac{\partial}{\partial y} f_1 = xe^y - \sin y$$

$$\frac{\partial}{\partial x} f_2 = 1 \text{ and } \frac{\partial}{\partial y} f_2 = 0$$

$$\frac{\partial}{\partial x} f_3 = 1 \text{ and } \frac{\partial}{\partial y} f_3 = e^y$$

$$Df = \begin{pmatrix} e^y & xe^y - \sin y \\ 1 & 0 \\ 1 & e^y \end{pmatrix}$$

Section 2.4

#3) This is a line in $\mathbb{R}^3$ that passes through $(-1,2,0)$ and $(1,3,1)$. To see this, note that this equation looks a lot like the parametrized line equation that we discussed in **Chapter 1**.

#8)

$$\vec{r'}(t) = \begin{pmatrix} 4e^t \\ 6(4t^3) \\ -\sin t \end{pmatrix} = \begin{pmatrix} 4e^t \\ 24t^3 \\ -\sin t \end{pmatrix}$$

#9)

$$\vec{c'}(t) = \begin{pmatrix} e^t \\ -\sin t \end{pmatrix}$$

Section 2.5

#2a)

$$\frac{\partial}{\partial x}f = 0 \text{ and } \frac{\partial}{\partial y}f = 0$$

Both partials are **continuous** $\Rightarrow$ f is **differentiable**.

#2e)

$$\frac{\partial}{\partial x}f = ye^{xy} \text{ and } \frac{\partial}{\partial y}f = xe^{xy}$$

Both partials are **continuous** $\Rightarrow$ f is **differentiable**.

#2g)

$$\frac{\partial}{\partial x}f = 4x^3 \text{ and } \frac{\partial}{\partial y}f = -4y^3$$

Both partials are **continuous** $\Rightarrow$ f is **differentiable**.

#4) Let's first use the chain rule to figure out what we want to verify. In order to see what we get for h_x, we'll define a function

$$g(x,y) = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$$

So now

$$h(x,y) = f(u(x,y), v(x,y)) = (f \circ g)(x,y)$$

By the chain rule,

$$\begin{pmatrix} h_x & h_y \end{pmatrix} = \nabla h = \nabla(f \circ g) = \nabla f \bullet Dg = \begin{pmatrix} f_u & f_v \end{pmatrix} \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

$$= \begin{pmatrix} f_u u_x + f_v v_x & f_u u_y + f_v v_y \end{pmatrix}$$

So we are trying to verify $h_x = f_u u_x + f_v v_x$.

$$h(x,y) = f(u(x,y), v(x,y)) = \frac{(e^{-x-y})^2 + (e^{xy})^2}{(e^{-x-y})^2 + (e^{xy})^2} = \frac{e^{-2x-2y} + e^{2xy}}{e^{-2x-2y} - e^{2xy}}$$

$$h_x = \frac{(e^{-2x-2y} - e^{2xy})\frac{\partial}{\partial x}(e^{-2x-2y} + e^{2xy}) - (e^{-2x-2y} + e^{2xy})\frac{\partial}{\partial x}(e^{-2x-2y} - e^{2xy})}{(e^{-2x-2y} - e^{2xy})^2}$$

$$= \frac{(e^{-2x-2y} - e^{2xy})(-2e^{-2x-2y} + 2ye^{2xy}) - (e^{-2x-2y} + e^{2xy})(-2e^{-2x-2y} - 2ye^{2xy})}{(e^{-2x-2y} - e^{2xy})^2}$$

$$\ldots = 4(1-y)\frac{e^{-2x-2y+2xy}}{(e^{-2x-2y} - e^{2xy})^2}$$

Remembering the definition of $u(x,y)$ and $v(x,y)$, we can write this as:

$$h_x = 4(1-y)\frac{u^2v^2}{(u^2-v^2)^2}$$

Now we will find the RHS (right hand side) of our chain rule identity:

$$f_u = \frac{(u^2-v^2)\frac{\partial}{\partial u}(u^2+v^2) - (u^2+v^2)\frac{\partial}{\partial u}(u^2-v^2)}{(u^2-v^2)^2}$$

$$= \frac{(u^2-v^2)(2u) - (u^2+v^2)(2u)}{(u^2-v^2)^2} = -4\frac{uv^2}{(u^2-v^2)^2}$$

$$f_v = \frac{(u^2-v^2)\frac{\partial}{\partial v}(u^2+v^2) - (u^2+v^2)\frac{\partial}{\partial v}(u^2-v^2)}{(u^2-v^2)^2}$$

$$= \frac{(u^2-v^2)(2v) - (u^2+v^2)(-2v)}{(u^2-v^2)^2} = 4\frac{u^2v}{(u^2-v^2)^2}$$

$$u_x = \frac{\partial}{\partial x}e^{-x-y} = -e^{-x-y} = -u$$

$$v_x = \frac{\partial}{\partial x}e^{xy} = ye^{xy} = yv$$

So,

$$f_uu_x + f_vv_x = -4\frac{uv^2}{(u^2-v^2)^2}(-u) + 4\frac{u^2v}{(u^2-v^2)^2}(yv)$$

$$= 4(1-y)\frac{u^2v^2}{(u^2-v^2)^2} = h_x$$

$\square$

#17a) To get $G(x, y(x))$ as a composition, we will define $h : \mathbb{R} \to \mathbb{R}^2$ as

$$h(x) = \begin{pmatrix} x \\ y(x) \end{pmatrix} \Rightarrow G(x, y(x)) = G \circ h$$

$$(G \circ h)(x) = 0 \Rightarrow \nabla(G \circ h) = \nabla(0) = 0$$

So by the chain rule, we get

$$0 = \nabla G \bullet Dh(x) = \begin{pmatrix} G_x & G_y \end{pmatrix} \begin{pmatrix} 1 \\ y' \end{pmatrix}$$

$$= G_x + G_y y' \Rightarrow y' = -\frac{G_x}{G_y}, G_y \neq 0$$

#17b) We will use a similar strategy as in #17a, but we now have to make some adjustments. $G : \mathbb{R}^3 \to \mathbb{R}^2$ so we will use $h : \mathbb{R} \to \mathbb{R}^3$:

$$G(x, y, z) = \begin{pmatrix} G_1(x, y, z) \\ G_2(x, y, z) \end{pmatrix} \text{ and } h(x) = \begin{pmatrix} x \\ y_1(x) \\ y_2(x) \end{pmatrix}$$

So,

$$0 = G(x, y_1(x), y_2(x)) = (G \circ h)(x) \Rightarrow \nabla(G \circ h) = 0$$

By the chain rule, we get:

$$DG \bullet Dh = \begin{pmatrix} \frac{\partial}{\partial x}G_1 & \frac{\partial}{\partial y}G_1 & \frac{\partial}{\partial z}G_1 \\ \frac{\partial}{\partial x}G_2 & \frac{\partial}{\partial y}G_2 & \frac{\partial}{\partial z}G_2 \end{pmatrix} \begin{pmatrix} 1 \\ y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x}G_1 + y_1'\frac{\partial}{\partial y}G_1 + y_2'\frac{\partial}{\partial z}G_1 \\ \frac{\partial}{\partial x}G_2 + y_1'\frac{\partial}{\partial y}G_2 + y_2'\frac{\partial}{\partial z}G_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} y_1'\frac{\partial}{\partial y}G_1 + y_2'\frac{\partial}{\partial z}G_1 \\ y_1'\frac{\partial}{\partial y}G_2 + y_2'\frac{\partial}{\partial z}G_2 \end{pmatrix} = \begin{pmatrix} -\frac{\partial}{\partial x}G_1 \\ -\frac{\partial}{\partial x}G_2 \end{pmatrix} \Rightarrow \begin{pmatrix} \frac{\partial}{\partial y}G_1 + \frac{\partial}{\partial z}G_1 \\ \frac{\partial}{\partial y}G_2 + \frac{\partial}{\partial z}G_2 \end{pmatrix} \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} -\frac{\partial}{\partial x}G_1 \\ -\frac{\partial}{\partial x}G_2 \end{pmatrix}$$

Using some linear algebra,

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial y}G_1 + \frac{\partial}{\partial z}G_1 \\ \frac{\partial}{\partial y}G_2 + \frac{\partial}{\partial z}G_2 \end{pmatrix}^{-1} \begin{pmatrix} -\frac{\partial}{\partial x}G_1 \\ -\frac{\partial}{\partial x}G_2 \end{pmatrix}$$

$$= \frac{1}{\frac{\partial}{\partial y}G_1\frac{\partial}{\partial z}G_2 - \frac{\partial}{\partial z}G_1\frac{\partial}{\partial y}G_2} \begin{pmatrix} \frac{\partial}{\partial z}G_2 & -\frac{\partial}{\partial z}G_1 \\ -\frac{\partial}{\partial y}G_2 & \frac{\partial}{\partial y}G_1 \end{pmatrix} \begin{pmatrix} -\frac{\partial}{\partial x}G_1 \\ -\frac{\partial}{\partial x}G_2 \end{pmatrix}$$

Or,

$$y_1' = \frac{\frac{\partial}{\partial z}G_1\frac{\partial}{\partial x}G_2 - \frac{\partial}{\partial x}G_1\frac{\partial}{\partial z}G_2}{\frac{\partial}{\partial y}G_1\frac{\partial}{\partial z}G_2 - \frac{\partial}{\partial z}G_1\frac{\partial}{\partial y}G_2}$$

$$y_2' = \frac{\frac{\partial}{\partial x}G_1\frac{\partial}{\partial y}G_2 - \frac{\partial}{\partial y}G_1\frac{\partial}{\partial x}G_2}{\frac{\partial}{\partial y}G_1\frac{\partial}{\partial z}G_2 - \frac{\partial}{\partial z}G_1\frac{\partial}{\partial y}G_2}$$

$$\text{for } \frac{\partial}{\partial y}G_1\frac{\partial}{\partial z}G_2 - \frac{\partial}{\partial z}G_1\frac{\partial}{\partial y}G_2 \neq 0$$

#17c)

$$y' = -\frac{\frac{\partial}{\partial x}(x^2+y^3+e^y)}{\frac{\partial}{\partial y}(x^2+y^3+e^y)} = -\frac{2x}{3y^2+e^y}$$

Section 2.6

#3b) The directional derivative is

$$\nabla f \bullet \vec{d}$$

where $\vec{d}$ is a unit vector.

$$\|\vec{d}\| = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3} \neq 1$$

So let

$$\vec{d_2} = \frac{\vec{d}}{\|\vec{d}\|} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$\nabla f = \begin{pmatrix} e^x & z & y \end{pmatrix}$$

$$\nabla f \bullet \vec{d_2} = \begin{pmatrix} e^x & z & y \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} = \frac{e^x}{\sqrt{3}} - \frac{z}{\sqrt{3}} + \frac{y}{\sqrt{3}}$$

$$(\nabla f \bullet \vec{d_2})(1,1,1) = \frac{e}{\sqrt{3}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} = \frac{e}{\sqrt{3}}$$

#4b)

$$f(x,y,z) = y^2 - x^2 \Rightarrow \nabla f = \begin{pmatrix} -2x & 2y & 0 \end{pmatrix}$$

$$\vec{x_0} = \begin{pmatrix} 1 \\ 2 \\ 8 \end{pmatrix} \Rightarrow \nabla f(\vec{x_0}) = (-2, 4, 0)$$

$$\nabla f(\vec{x_0}) \bullet (\vec{x} - \vec{x_0}) = 0 \Rightarrow -2(x-1) + 4(y-2) + 0(z-8) = 0$$

$$\text{or } -(x-1) + 2(y-2) = 0$$

$$\text{or } -x + 2y - 3 = 0$$

#4c)

$$f(x,y,z) = xyz \Rightarrow \nabla f = \begin{pmatrix} yz & xz & xy \end{pmatrix}$$

$$\vec{x_0} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow \nabla f(\vec{x_0}) = (1,1,1)$$

$$\nabla f(\vec{x_0}) \bullet (\vec{x} - \vec{x_0}) = 0 \Rightarrow (x-1) + (y-1) + (z-1) = 0$$

$$\text{or } x + y + z - 3 = 0$$

#19) Let us look at some even function that we know:

$$f(x) = \cos x \Rightarrow \nabla f(0) = f'(0) = -\sin 0 = 0$$

$$f(x) = x^2 \Rightarrow \nabla f(0) = f'(0) = 2(0) = 0$$

Let's try a multivariable example:

$$f(x,y) = \sin x \sin y \Rightarrow \nabla f(\vec{0}) = \begin{pmatrix} \cos 0 \sin 0 + \sin 0 \cos 0 \end{pmatrix} = \vec{0}$$

So our guess would be that $\nabla f(\vec{0}) = \vec{0}$.
This is one of the proofs listed on the website.