Math 212: Answers to Assignment 3

Section 3.1

#3) We use the chain rule here

$$\frac{\partial}{\partial x}f = \frac{\partial}{\partial x}(\cos(xy^2)) = (-\sin(xy^2))(y^2) = -y^2\sin(xy^2)$$

$$\frac{\partial}{\partial y}f = \frac{\partial}{\partial y}(\cos(xy^2)) = (-\sin(xy^2))(2xy) = -2xy\sin(xy^2)$$

$$\frac{\partial^2}{\partial x^2}f = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}f\right) = \frac{\partial}{\partial x}(-y^2\sin(xy^2))$$

$$= -y^2(\cos(xy^2))(y^2) = -y^4\cos(xy^2)$$

Now we use the product rule as well

$$\frac{\partial^2}{\partial y^2}f = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}f\right) = \frac{\partial}{\partial y}(-2xy\sin(xy^2))$$

$$= \sin(xy^2)\frac{\partial}{\partial y}(-2xy) - 2xy\frac{\partial}{\partial y}(\sin(xy^2))$$

$$= -2x\sin(xy^2) - 4x^2y^2\cos(xy^2)$$

$$\frac{\partial^2}{\partial y\partial x}f = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}f\right) = \frac{\partial}{\partial y}(-y^2\sin(xy^2))$$

$$= \sin(xy^2)\frac{\partial}{\partial y}(-y^2) - y^2\frac{\partial}{\partial y}(\sin(xy^2)) = -2y\sin(xy^2) - 2xy^3\cos(xy^2)$$

$$\frac{\partial^2}{\partial x\partial y}f = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}f\right) = \frac{\partial}{\partial x}(-2xy\sin(xy^2))$$

$$= \sin(xy^2)\frac{\partial}{\partial x}(-2xy) - 2xy\frac{\partial}{\partial x}(\sin(xy^2)) = -2y\sin(xy^2) - 2xy^3\cos(xy^2)$$

So $\frac{\partial^2}{\partial x\partial y}f = \frac{\partial^2}{\partial y\partial x}f$.

#6) Chain rule

$$\frac{\partial}{\partial x}f = \frac{1}{x-y}\frac{\partial}{\partial x}(x-y) = \frac{1}{x-y}$$

$$\frac{\partial}{\partial y}f = \frac{1}{x-y}\frac{\partial}{\partial y}(x-y) = -\frac{1}{x-y}$$

$$\frac{\partial^2}{\partial x^2}f = \frac{\partial}{\partial x}\left(\frac{1}{x-y}\right) = -\frac{1}{(x-y)^2}$$

$$\frac{\partial^2}{\partial y^2}f = \frac{\partial}{\partial y}\left(-\frac{1}{x-y}\right) = -\frac{1}{(x-y)^2}$$

$$\frac{\partial^2}{\partial y\partial x}f = \frac{\partial}{\partial y}\left(\frac{1}{x-y}\right) = \frac{1}{(x-y)^2}$$
$$\frac{\partial^2}{\partial x\partial y}f = \frac{\partial}{\partial x}\left(-\frac{1}{x-y}\right) = \frac{1}{(x-y)^2}$$

So $\frac{\partial^2}{\partial y\partial x}f = \frac{\partial^2}{\partial x\partial y}f$.

#12)

$$\frac{\partial^3}{\partial x\partial y\partial z}(ze^{xy} + yz^3x^2) = \frac{\partial^2}{\partial x\partial y}(e^{xy} + 3yz^2x^2)$$

$$= \frac{\partial}{\partial x}(xe^{xy} + 3z^2x^2) = e^{xy} + xye^{xy} + 6z^2x.$$

$$\frac{\partial^3}{\partial z \partial y \partial x}(ze^{xy} + yz^3x^2) = \frac{\partial^2}{\partial z \partial y}(zye^{xy} + 2yz^3x)$$
$$= \frac{\partial}{\partial z}(ze^{xy} + xyze^{xy} + 2z^3x) = e^{xy} + xye^{xy} + 6z^2x.$$

So the two mixed partials are equal. □

Section 3.3

#1) We need to find solutions to $f_x = 0$ and $f_y = 0$ to find critical points of f.

$$f_x = 2x + y, \text{ and } f_y = x - 2y$$

$$f_x = 0 \Rightarrow y = -2x, \text{ and } f_y = 0 \Rightarrow x - 2(-2x) = 0, \text{ or } x = y = 0$$

So our only critical point is $(0, 0)$. Now we need to determine if this point is a maximum, minimum, or saddle point.

$$f_{xx} = 2 > 0$$

$$f_{xx}f_{yy} - f_{xy}^2 = 2(-2) - 1^2 = -5 < 0$$

The failure of part 3 of the second derivative test tells us that this is a saddle point. $\square$

#10)

$$f_x = \sin y, f_y = 1 + x\cos y$$

Now we test for critical points:

$$f_x = 0 \Rightarrow y = n\pi \text{ for } n \in \mathbb{Z}$$

Now $f_y = 1 + x\cos(n\pi) = 1 + x(-1)^n$. So $f_y = 0 \Rightarrow x = (-1)^{n+1}$

So our critical points are $((-1)^{n+1}, n\pi)$ for all integers n. Now we see what kind of extrema they are:

$$f_{xx} = 0, f_{yy} = -x\sin y, f_{xy} = -\cos y$$

$$\text{So } f_{xx}f_{yy} - f_{xy}^2 = 0 - \cos^2 y < 0 \text{ for the above points}$$

So all of our points are saddle points. $\square$

#23) We see that we have to maxmize f as follows

$$f(x, y, z) = 2xy + 2xz + 2yz, \text{ for } xyz = V$$

where V is a constant. We will now use our "trick" to solve for z in terms of x and y.

$$z = \frac{V}{xy} \Rightarrow \widetilde{f}(x, y) = f(x, y, \frac{V}{xy}) = 2xy + \frac{2V}{x} + \frac{2V}{y}$$

Now our problem is to maximize $\widetilde{f}$ of the region $\{x > 0, y > 0\}$ which is a region with no boundary. So we will use the Second Derivate Test.

$$\widetilde{f_x} = 2y - \frac{2V}{x^2}, \widetilde{f_y} = 2x - \frac{2V}{y^2}$$

$$\widetilde{f_x} = 0 \Rightarrow y = \frac{V}{x^2}$$

Substituting into $\widetilde{f_y} = 0$ yields:

$$2x - 2V\frac{x^4}{V^2} = 2x - 2\frac{x^4}{V} = 0 \Rightarrow x = 0 \text{ or } x = V^{\frac{1}{3}}$$

Since $x > 0$ is part of the problem, we are left with $x = y = z = V^{\frac{1}{3}}$.

$$\widetilde{f_{xx}} = \frac{4V}{x^3} = 4, \widetilde{f_{yy}} = \frac{4V}{y^3} = 4$$

$$\widetilde{f_{xy}} = 2$$

$$\text{So } f_{xx} > 0, \text{ and } f_{xx}f_{yy} - f_{xy}^2 = 16 - 4 = 12 > 0$$

So this point (when our box is a cube) is a minimum.

Section 3.4

#1)

$$f_x = 1, f_y = -1, f_z = 1$$

$$g_x = 2x, g_y = 2y, g_z = 2z$$

$$\text{So } \nabla f = \lambda \nabla g \Rightarrow 1 = \lambda 2x, -1 = \lambda 2y, \text{ and } 1 = \lambda 2z$$

Since $\lambda \neq 0$ (otherwise the equations couldn't be satisfied), we have that $x = -y = z$. So

$$g(x, y, z) = g(x, -x, x) = 2 \Rightarrow x = \pm\sqrt{\frac{2}{3}}$$

So we have two points:

$$\vec{p_1} = (\sqrt{\frac{2}{3}}, -\sqrt{\frac{2}{3}}.\sqrt{\frac{2}{3}}), \text{ and } \vec{p_2} = (-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}, -\sqrt{\frac{2}{3}})$$

$$f(\vec{p_1}) = 3\sqrt{\frac{2}{3}} = \sqrt{6}, f(\vec{p_2}) = -3\sqrt{\frac{2}{3}} = -\sqrt{6}$$

So $\vec{p_1}$ is the maximum and $\vec{p_2}$ is the minimum. $\square$

#3)

$$f_x = 1, f_y = 0, g_x = 2x, g_y = 4y$$

So we solve

$$1 = \lambda 2x$$

$$0 = \lambda 4y$$

Since $\lambda = 0$ would make this system unsolvable, the second equation yeilds $y = 0$. Plugging this into $g(x, y) = 3$ gives us that $x = \pm\sqrt{3}$. So we have two points:

$$f(\sqrt{3}, 0) = \sqrt{3}, f(-\sqrt{3}, 0) = -\sqrt{3}$$

So $(\sqrt{3}, 0)$ is the maximum, and $(\sqrt{3}, 0)$ is the minimum. $\square$

#5)

$$\nabla f = \lambda \nabla g \Rightarrow 3 = \lambda 4x, 2 = \lambda 6y$$

Noting again that $\lambda, x, y \neq 0$,

$$\lambda = \frac{3}{4x} \Rightarrow 2 = \frac{3}{4x} 6y \Rightarrow x = \frac{9}{4} y$$

$$g(x,y) = g\left(\frac{9}{4}y, y\right) = 2\frac{81}{16}y^2 + 3y^2 = 3 \Rightarrow y = \pm 2\sqrt{\frac{2}{35}} = \pm\frac{4}{\sqrt{70}}$$

So we get the points $(\pm\frac{9}{\sqrt{70}}, \pm\frac{4}{\sqrt{70}})$. Evaluating f at these points tells us that the positive choice is the maximum. The negative is the minimum. $\square$

#13) Let

$$f(r,h) = 2\pi r^2 + 2\pi rh, \;\; g(r,h) = \pi r^2 h$$

So in this question, we are asked to maximize $f(r,h)$ over $g(r,h) = 1$ where r is the radius of the top and h is the height of the can. Since $g(r,h) = 1$ is not a bounded set in $\mathbb{R}^2$, we need to solve for one variable in terms of the other to eliminate our $g(r,h) = 1$ condition.

$$\pi r^2 h = 1 \Rightarrow h = \frac{1}{\pi r^2}$$

So now,

$$\widetilde{f}(r) = 2\pi r^2 + \frac{2}{r}$$

We can now solve this with single variable calculus

$$\widetilde{f}'(r) = 4\pi r - \frac{2}{r^2} = 0$$

$$\Rightarrow 4\pi r^3 - 2 = 0, (r \neq 0) \Rightarrow r^3 = \frac{1}{2\pi}$$

$$\Rightarrow r = \sqrt[3]{\frac{1}{2\pi}}$$

$$\widetilde{f}''(r) = 4\pi + \frac{4}{r^3} = 12\pi > 0, \text{ for } r = \sqrt[3]{\frac{1}{2\pi}}$$

So when we have a minimum when we make the can with radius $\sqrt[3]{\frac{1}{2\pi}}$ and height $\sqrt[3]{\frac{4}{\pi}}$. $\square$

Section 4.1

#19a) When we have a curve defined by a graph of y(x), we can always use the path $\vec{c}(t) = \big(t \quad y(t)\big)$, so we have

$$\vec{c}(t) = \big(t \quad e^t\big)$$

#19b) When we see a curve defined by $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = c^2$, we can use the path $\vec{c}(t) = \big(ac\cos t \quad bc\sin t\big)$. So we can use $a = \frac{1}{2}, b = c = 1$ to get

$$\vec{c}(t) = \left(\tfrac{\cos t}{2} \quad \sin t\right)$$

#19c) We have been given in Chapter 1 a parametrization for a line, given to points. So we'll use this point-point form $\vec{c}(t) = \vec{p_0} + t(\vec{p_1} - \vec{p_0})$

$$\vec{c}(t) = \big(at \quad bt \quad ct\big)$$

#19d) Similarly to #19b, we have

$$\vec{c}(t) = \left(\tfrac{2}{3}\cos t \quad \tfrac{1}{2}\sin t\right)$$

Section 4.2

#1)

$$\vec{c'}(t) = \begin{pmatrix}-2\sin t & 2\cos t & 1\end{pmatrix}$$

$$\int_0^{2\pi} \sqrt{(-2\sin t)^2 + (2\cos t)^2 + 1^2}dt$$

$$= \int_0^{2\pi} \sqrt{4\sin^2 t + 4\cos^2 t + 1}dt$$

$$= \int_0^{2\pi} \sqrt{5}dt = 2\pi\sqrt{5}$$

$\square$

#2)

$$\vec{c'}(t) = \begin{pmatrix}0 & 6t & 3t^2\end{pmatrix}$$

$$\int_0^1 \sqrt{0^2 + (6t)^2 + (3t^2)^2}dt = \int_0^1 \sqrt{36t^2 + 9t^4}dt$$

$$= \int_0^1 3t\sqrt{4+t^2}dt, \text{ since } t \geq 0$$

$$= \frac{3}{2}\int_0^1 2t\sqrt{4+t^2}dt$$

$$= \frac{3}{2}\int_4^5 \sqrt{u}du = \frac{3}{2}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_{u=4}^{u=5}$$

$$= 5\sqrt{5} - 8$$

$\square$

#3)

$$\vec{c'}(t) = \begin{pmatrix}3\cos 3t & -3\sin 3t & 3t^{\frac{1}{2}}\end{pmatrix}$$

$$\int_0^1 \sqrt{(3\cos 3t)^2 + (-3\sin 3t)^2 + (3t^{\frac{1}{2}})^2}dt$$

$$= \int_0^1 \sqrt{9\cos^2 3t + 9\sin^2 3t + 9t}dt = \int_0^1 3\sqrt{1+t}dt$$

$$= \left[2(1+t)^{\frac{3}{2}}\right]_{t=0}^{t=1} = 4\sqrt{2} - 2 \quad \square$$

#8)

$$\vec{c}'(t) = \begin{pmatrix}1 & \sin t + t\cos t & \cos t - t\sin t\end{pmatrix}$$

$$\int_0^{\pi} \sqrt{1^2 + (\sin t + t\cos t)^2 + (\cos t - t\sin t)^2} dt$$

$$= \int_0^{\pi} \sqrt{1 + \sin^2 t + 2t\sin t\cos t + t^2\cos^2 t + \cos^2 t - 2t\sin t\cos t + t^2\sin^2 t}\ dt$$

$$= \int_0^{\pi} \sqrt{2+t^2}\ dt = \left[\frac{t}{2}\sqrt{2+t^2} + \log\left|t + \sqrt{2+t^2}\right|\right]_{t=0}^{t=\pi}$$

$$= \frac{\pi}{2}\sqrt{2+\pi^2} + \log\left|\pi + \sqrt{2+\pi^2}\right| - \log\left|\sqrt{2}\right|$$

$$= \frac{\pi}{2}\sqrt{2+\pi^2} + \log\left|\frac{\pi}{\sqrt{2}} + \sqrt{1 + \frac{\pi^2}{2}}\right| \quad \square$$

#9)

$$\vec{c}'(t) = \begin{pmatrix}2 & 2t & \frac{1}{t}\end{pmatrix}$$

$$\int_1^2 \sqrt{2^2 + (2t)^2 + (\frac{1}{t})^2} dt$$

$$= \int_1^2 \sqrt{4 + 4t^2 + \frac{1}{t^2}} dt = \int_1^2 \sqrt{\left(2t + \frac{1}{t}\right)^2}\ dt$$

$$= \int_1^2 2t + \frac{1}{t} dt = \left[t^2 + \log t\right]_{t=1}^{t=2}$$

$$= 4 + \log 2 - 1 = 3 + \log 2 \quad \square$$