Math 212: Answers to Assignment 5

Section 5.1

#6a) By **Cavaleri's Principle** we know that

$$Volume = \int_a^b A(x)dx$$

Where $A(x_0)$ is the area of the cross section of the body with the $x = x_0$ plane. In this case our cross section is just a circle of radius $f(x_0)$

$$\Rightarrow A(x) = \pi[f(x)]^2 \Rightarrow Vol = \int_a^b \pi[f(x)]^2 dx = \pi \int_a^b [f(x)]^2 dx \quad \square$$

#6b) By part a,

$$Vol = \pi \int_{-1}^{3} (-x^2 + 2x + 3)^2 dx = \pi \int_{-1}^{3} (x-3)^2(x+1)^2 dx$$

Using $u = x + 1$, we get

$$= \pi \int_0^4 (u-4)^2 u^2 du = \pi \int_0^4 u^4 - 8u^3 + 16u^2 du$$

$$= \pi \left[\frac{u^5}{5} - 2u^4 + \frac{16u^3}{3} \right]_{u=0}^{u=4} = \pi \left(\frac{1024}{5} - 512 + \frac{1024}{3} \right) = \pi \frac{512}{15} \quad \square$$

<u>Section 5.2</u>

#2b)

$$\int_0^1 \int_0^1 (ax + by + c)dxdy = \int_0^1 \left[\frac{ax^2}{2} + bxy + cx\right]_{x=0}^{x=1} dy$$

$$= \int_0^1 \left(\frac{a}{2} + by + c\right) dy = \left[\frac{ay}{2} + \frac{by^2}{2} + cy\right]_{y=0}^{y=1}$$

$$= \frac{a}{2} + \frac{b}{2} + c \quad \square$$

#2c)

$$\int_0^1 \int_0^1 \sin(x + y)dxdy = \int_0^1 [-\cos(x + y)]_{x=0}^{x=1} dy$$

$$= \int_0^1 \cos y - \cos(y + 1)dy = [\sin y - \sin(y + 1)]_{y=0}^{y=1}$$

$$= (\sin 1 - \sin 2) - (\sin 0 - \sin 1) = 2\sin 1 - \sin 2 \quad \square$$

<u>Section 5.3</u>

#2b) Since $|x| = x$ for $x \geq 0$ and $|x| = -x$ for $x < 0$ we have to split the integral into two parts:

$$\int_{-1}^{1}\int_{-2|x|}^{|x|} e^{x+y}dydx = \int_{-1}^{0}\int_{2x}^{-x} e^{x+y}dydx + \int_{0}^{1}\int_{-2x}^{x} e^{x+y}dydx$$

$$= \int_{-1}^{0}\left[e^{x+y}\right]_{y=2x}^{y=-x} dx + \int_{0}^{1}\left[e^{x+y}\right]_{y=-2x}^{y=x} dx = \int_{-1}^{0}\left(1-e^{3x}\right)dx + \int_{0}^{1}\left(e^{2x}-e^{-x}\right)dx$$

$$= \left[x - \frac{e^{3x}}{3}\right]_{x=-1}^{x=0} + \left[\frac{e^{2x}}{2} + e^{-x}\right]_{x=0}^{x=1} = 0 - \frac{1}{3} + 1 + \frac{e^{-3}}{3} + \frac{e^{2}}{2} + e^{-1} - \frac{1}{2} - 1$$

$$= \frac{e^{-3}}{3} + \frac{e^{2}}{2} + e^{-1} - \frac{5}{6} \quad \square$$

#2d)

$$\int_{0}^{\frac{\pi}{2}}\int_{0}^{\cos x}(y\sin x)dydx = \int_{0}^{\frac{\pi}{2}}\left[\frac{y^2}{2}\sin x\right]_{y=0}^{y=\cos x} dx$$

$$= \frac{1}{2}\int_{0}^{\frac{\pi}{2}}(\cos^2 x\sin x)dx = \frac{1}{2}\left[-\frac{\cos^3 x}{3}\right]_{x=0}^{x=\frac{\pi}{2}} = \frac{1}{6} \quad \square$$

#2e)

$$\int_{0}^{1}\int_{y^2}^{y}(x^n + y^m)dxdy = \int_{0}^{1}\left[\frac{x^{n+1}}{n+1} + xy^m\right]_{x=y^2}^{x=y} dy$$

$$= \int_{0}^{1}\left(\frac{y^{n+1}}{n+1} + y^{m+1} - \frac{y^{2n+2}}{n+1} - y^{m+2}\right)dy$$

$$= \left[\frac{1}{n+1}\frac{y^{n+2}}{n+2} + \frac{y^{m+2}}{m+2} - \frac{1}{n+1}\frac{y^{2n+3}}{2n+3} - \frac{y^{m+3}}{m+3}\right]_{y=0}^{y=1}$$

$$= \frac{1}{(n+1)(n+2)} + \frac{1}{m+2} - \frac{1}{(n+1)(2n+3)} - \frac{1}{m+3} \quad \square$$

#6) We'll only set up the integral (the rest of the work is long and not very enlightening, plus, I didn't grade the final result):

$$\int_0^{\frac{10}{3}} \int_0^{\frac{5}{2}-\frac{3}{4}x} (x^2+y^2)dydx$$

or

$$\int_0^{\frac{5}{2}} \int_0^{\frac{10}{3}-\frac{4}{3}y} (x^2+y^2)dxdy \quad \square$$

#16) Using $T(u,v) = \begin{pmatrix} a_1u + b_1v & a_2u + b_2v \end{pmatrix}$ and $D^* = [0,1] \times [0,1]$, we get

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

So by the **Change of Variables Theorem**,

$$A(D) = \iint_D dA = \iint_{D^*} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv = |a_1b_2 - a_2b_1| \quad \square$$

#4)

$$\max_{0\le x,y\le 1}\left\{\frac{\sin x}{1+(xy)^4}\right\} \le \frac{\sin 1}{1} \le 1$$

So

$$\iint_{[0,1]^2=[0,1]\times[0,1]} \frac{\sin x}{1+(xy)^4}dxdy \le 1\cdot 1 = 1$$

Also

$$\frac{\sin x}{1+(xy)^4} \ge \frac{\sin x}{2},\ \text{for } 0\le x,y\le 1$$

So

$$\iint_{[0,1]^2} \frac{\sin x}{1+(xy)^4}dxdy \ge \iint_{[0,1]^2} \frac{\sin x}{2}dydx$$

$$= \int_0^1 \frac{\sin x}{2}dx = \left[-\frac{\cos x}{2}\right]_0^1 = \frac{1-\cos 1}{2}$$

Therefore

$$\frac{1}{2}(1-\cos 1) \le \iint_{[0,1]^2} \frac{\sin x}{1+(xy)^4}dxdy \le 1 \quad \square$$

#10)

$$\iint_D e^{x-y}dxdy = \int_0^1\int_x^{3x} e^{x-y}dydx + \int_1^2\int_x^{4-x} e^{x-y}dydx$$

$$= \int_0^1 \left[-e^{x-y}\right]_{y=x}^{y=3x} dx + \int_1^2 \left[-e^{x-y}\right]_{y=x}^{y=4-x} dx$$

$$= \int_0^1 (1-e^{-2x})dx + \int_1^2 (1-e^{2x-4})dx = \left[x+\frac{e^{-2x}}{2}\right]_0^1 + \left[x-\frac{e^{2x-4}}{2}\right]_1^2$$

$$= 1+\frac{e^{-2}}{2} - 0 - \frac{1}{2} + 2 - \frac{1}{2} - 1 + \frac{e^{-2}}{2} = \frac{e^{-2}}{2} + 1 \quad \square$$

<u>Section 5.5</u>

#8)

$$D = \{0 \le x \le 4, 0 \le y \le 4 - x, 0 \le z \le x + y + 1\} \quad \square$$

#18)

$$\int_0^2 \int_0^x \int_0^{x+y} dzdydx = \int_0^2 \int_0^x (x+y)dydx$$

$$= \int_0^2 \left[xy + \frac{y^2}{2} \right]_{y=0}^{y=x} dx = \int_0^2 \frac{3}{2}x^2 dx$$

$$= \left[\frac{3}{2}\frac{x^3}{3} \right]_0^2 = \frac{3}{2} \cdot \frac{8}{3} = 4 \quad \square$$

#22b)

$$\int_0^1 \int_z^1 \int_y^1 f(x,y,z)dxdydz \quad \square$$