Vectors, Matrices, and You

A Quick Overview of Chapter 1

Math 212

Vectors, Dot Product, and the Norm

Just like a regular variable is an element of $\mathbb{R}$, a **(real) vector** is a general element of $\mathbb{R}^n$. A vector $\vec{v}$ is defined as follows:

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \text{ or } \vec{v} = (v_1, v_2, ..., v_n)$$

There are special notations for vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$:

$$\vec{v} = (x, y) \text{ and } \vec{v} = (x, y, z)$$

to represent the coordinates along the **x-axis**,**y-axis** and **z-axis**.

Vectors have a component-wise addition operation:

$$\vec{v} + \vec{w} = (v_1 + w_1, v_2 + w_2, ..., v_n + w_n)$$

They also have a component-wise scalar multiplication:

$$\alpha\vec{v} = (\alpha v_1, \alpha v_2, ..., \alpha v_n)$$

There are **standard basis vectors** in $\mathbb{R}^3$:

$$\vec{i} = (1, 0, 0), \vec{j} = (0, 1, 0), \vec{k} = (0, 0, 1)$$

This allows us an alternative way to write a vector in $\mathbb{R}^3$:

$$\vec{v} = (x, y, z) = x\vec{i} + y\vec{j} + z\vec{k}$$

There is a dot product operation $\bullet : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

$$\vec{v} \bullet \vec{w} = v_1 \cdot w_1 + v_2 \cdot w_2 + ... + v_n \cdot w_n$$

with the following properties:

1. $\vec{v} \bullet \vec{v} \geq 0$, where $\vec{v} \bullet \vec{v} = 0 \Leftrightarrow \vec{v} = \vec{0}$
2. $\alpha(\vec{v} \bullet \vec{w}) = (\alpha\vec{v}) \bullet \vec{w} = \vec{v} \bullet (\alpha\vec{w})$
3. $\vec{v} \bullet (\vec{w} + \vec{u}) = \vec{v} \bullet \vec{w} + \vec{v} \bullet \vec{u}$ and $(\vec{v} + \vec{w}) \bullet \vec{u} = \vec{v} \bullet \vec{u} + \vec{w} \bullet \vec{u}$
4. $\vec{v} \bullet \vec{w} = \vec{w} \bullet \vec{v}$

So, given this dot product, we define the **norm** of a vector $\vec{v}$ as

$$\left\|\vec{v}\right\| = \sqrt{\vec{v} \bullet \vec{v}}$$

When we refer to a vector as **normalized**, we mean that it has norm of 1. To normalize a given vector, we take $\vec{v}/\left\|\vec{v}\right\|$. The norm also gives us a formula for distance between vectors:

$$dist(\vec{v}, \vec{w}) = \left\|\vec{v} - \vec{w}\right\|$$

Also, if θ is the angle between vectors $\vec{v}$ and $\vec{w}$, then

$$\vec{v} \bullet \vec{w} = \left\|\vec{v}\right\| \left\|\vec{w}\right\| \cos\theta$$

(Note that this equation holds for any n-dimensional vector, not just for $\mathbb{R}^2$ or $\mathbb{R}^3$)

We have the **Cauchy-Bunyakovsky-Schwarz inequality**

$$\vec{v} \bullet \vec{w} \leq \left\|\vec{v}\right\| \cdot \left\|\vec{w}\right\|$$

The **othogonal projection** of $\vec{v}$ onto $\vec{a}$ is

$$\vec{v_a} = \frac{\vec{a} \bullet \vec{v}}{\left\|\vec{a}\right\|^2}\vec{a}$$

The norm also holds to the **triangle inequality**

$$\left\|\vec{v} + \vec{w}\right\| \leq \left\|\vec{v}\right\| + \left\|\vec{w}\right\|$$

Matrices, Determinant, and the Cross Product

An m by n **(real) matrix** is an array of m and rows and n columns and is an element of $\mathbb{R}^m \times \mathbb{R}^n$. A matrix is written as

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix} \text{ or } A = \big(a_{i,j}\big)$$

The **determinant** of a **square** $(n \times n)$ matrix is as follows:

$$Det(A) = \begin{vmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{vmatrix} = \sum_{j=1}^{n} (-1)^{k+j} a_{k,j} Det(A_{k,j})$$

$$= \sum_{i=1}^{m} (-1)^{i+l} a_{i,l} Det(A_{i,l}), \text{ for any } 0 \leq k \leq m, 0 \leq l \leq n$$

The notation $A_{i,j}$ means the matrix made by elimating both the i^{th} row and j^{th} column. For example, in the 3×3 case:

$$Det \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} = a_{1,1} Det \begin{pmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{pmatrix} - a_{1,2} Det \begin{pmatrix} a_{2,1} & a_{2,3} \\ a_{3,1} & a_{3,3} \end{pmatrix} + a_{1,3} Det \begin{pmatrix} a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{pmatrix}$$

The determinant has the following properties:

1. Swapping two rows or columns changes the sign of the determinant.
2. Adding a multiple of one row or column to another doesn't affect the sign.
3. Multiplying a row or column by a scalar also multiplies the determinant.

The **cross-product** is a special operation, $\times : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$

$$\vec{v} \times \vec{w} = Det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}$$

Unlike the dot product, the cross product only applies to 3-dimensional vectors. The cross product has the following properties:

1. $\|\vec{v} \times \vec{w}\| = \|\vec{v}\| \cdot \|\vec{w}\| \sin\theta$, where θ is the angle between $\vec{v}$ and $\vec{w}$
2. $\vec{v} \times \vec{w} \perp \vec{v}$ and $\vec{v} \times \vec{w} \perp \vec{w}$
3. $\vec{v} \times \vec{w} = 0 \Leftrightarrow \vec{v}$ or $\vec{w} = 0$ or $\vec{v} = \alpha\vec{w}$ for scalar α

4. $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$

5. $v \times (\vec{w} + \vec{u}) = \vec{v} \times \vec{w} + \vec{v} \times \vec{u}$ and $(\vec{v} + \vec{w}) \times u = \vec{v} \times \vec{u} + \vec{w} \times \vec{u}$

6. $(\alpha\vec{v}) \times \vec{w} = \alpha(\vec{v} \times \vec{w}) = v \times (\alpha\vec{w})$ for scalar α

We are able to multiply certain matrices together, but when two matrices are square and of the same size, we can always multiply them according to the following rule:

$$AB = \left(a_{i,j}\right)\left(b_{i,j}\right) = \left(c_{i,j}\right) = C \text{ where } c_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j}$$

In general, this operation is not **commutative**, i.e.

$$AB \neq BA \text{ for certain matrices } A \text{ and } B$$

A special matrix called the **identiy** matrix, I_n, an $n \times n$ matrix such that

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \text{ and for any } n \times n \text{ matrix } A, AI_n = I_n A = A$$

(We usually omit the n in cases where we know the dimension.)

$$\text{Square matrix } A \text{ is } \textbf{invertible } := \exists A^{-1} \ni A \cdot A^{-1} = A^{-1} \cdot A = I$$

We have another important property of invertibility given by the determinant

$$\text{Square matrix } A \text{ is } \textbf{invertible } \Leftrightarrow Det(A) \neq 0$$

One final property of interest concerning matrix multiplication is $Det(AB) = Det(A)Det(B)$.

A Little Geometry, Parametrization, and Other Coordinate Systems

In $\mathbb{R}^2$, we have the usual equations for a line

$$y = mx + b \text{ and } ax + by + c = 0$$

In any dimension, we can **parametrize** a line going through point $\vec{p}$ in direction $\vec{v}$ by

$$\vec{\ell}(t) = \vec{p} + t\vec{v}$$

If we want to make an equation for a line passing through points $\vec{p_1}$ and $\vec{p_2}$, we use the **point-point form**

$$\vec{\ell}(t) = \vec{p_1} + t(\vec{p_2} - \vec{p_1})$$

Since this form has the properties $\vec{\ell}(0) = \vec{p_1}$ and $\vec{\ell}(1) = \vec{p_2}$, this will be a useful form in this class.

In $\mathbb{R}^3$, we define a plane going through a point $\vec{p} = (x_0, y_0, z_0)$ with **normal vector** $\vec{n} = (a, b, c)$ as

$$P := a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Using $\vec{n}$ and $\vec{p}$ as above, we can find the distance from point $\vec{a} = (x_1, y_1, z_1)$ as

$$Dist(\vec{a}, P) = \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}}$$

The coordinate system we usually use, involving the **x-axis** and such, is called the **Cartesian Coordinate System**. There are other systems that will be used in this course. The **Polar Coordinate System** (for $\mathbb{R}^2$) is defined by the following identities

$$x = r\cos\theta$$
$$y = r\sin\theta$$

Where $r \geq 0$ and $0 \leq \theta \leq 2\pi$.

The **Cylindrical Coordinate System** (for $\mathbb{R}^3$) is defined by the following identities

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

Where $r \geq 0$ and $0 \leq \theta \leq 2\pi$. This is essentially replacing (x, y) with polar coordinates and leaving z as is.

The last standard system we'll consider is the **Spherical Coordinate System**.

$$x = \rho \cos\theta \sin\phi$$
$$y = \rho \sin\theta \sin\phi$$
$$z = \rho \cos\phi$$

Where $\rho \geq 0$, $0 \leq \theta \leq 2\pi$, and $0 \leq \phi \leq \pi$.