# Geometric Topology with Andrew Putman

transcribed by Jack Petok

## Contents

# 1 8/25/2015: Manifolds

Perhaps as an undergraduate, you had a general topology course where you named some properties of spaces and studied pathologies where these nice properties failed. Such spaces are not the topic of this class. In particular, we will be studying a nice class of spaces called **manifolds**.

**Definition 1.** A **manifold of dimension** $n$ is a Hausdorff, paracompact space $M^n$ such that, for every $p \in M^n$, there exists a **chart** $(U, \varphi)$; that is, $U \subset M^n$ is an open neighborhood of $p$ together with a homeomorphism $\varphi\colon U \to V \subset \mathbf{R}^n$.

We will probably never use the words Hausdoff or paracompact in this course again.

## 1.1 Smooth manifolds

Something we'd like to do on manifolds is calculus. Certainly one can do calculus in a chart of $M^n$. But there is a problem trying to globalize calculus to the manifold; namely, we might have two different charts around $p \in M^n$ which are incompatible in the sense that they disagree as to which functions are smooth. In order to do calculus, our manifold needs to be equipped with some global smooth structure.

**Definition 2.** Given two charts $\varphi_1\colon U_1 \to V_1, \varphi_2\colon U_2 \to V_2$, the **transition function** $\tau_{12}$ is the function

$$\tau_{12}\colon \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$$

$$\tau_{12} := \varphi_2 \circ \varphi_1^{-1}$$

**Definition 3.** Let $I$ be a set. A **smooth atlas** $\mathcal{A}$ **indexed by** $I$ on a manifold $M^n$ is a set of charts

$$\{\varphi_i\colon U_i \to V_i\}_{i \in I}$$

such that

(1) $\{U_i\}_{i \in I}$ covers $M^n$.

(2) All transition functions are smooth.

Two smooth atlases $\mathcal{A}_1, \mathcal{A}_2$ on $M^n$ are **compatible** if $\mathcal{A}_1 \cup \mathcal{A}_2$ is a smooth atlas.

It is easy to check that compatibility is an equivalence relation on atlases. This leads us to make the following definition.

**Definition 4.** A **smooth manifold** is a manifold equipped with an equivalence class of smooth atlases [1].

**Example 1.** Let $U \subset \mathbf{R}^n$ be an open set. Then $U$ is a naturally smooth manifold, as demonstrated by the atlas with the single chart

$$\mathrm{id}\colon U \to V = U.$$

This may seem silly, but one can make an entire career out of studying such manifolds! Let $K \subset \mathbf{R}^3$ be a knot (an embedding of $S^1$ into $\mathbf{R}^3$). Then the knot complement $K \backslash \mathbf{R}^3$ is a smooth manifold in this way.

More generally, if $M^n$ is a smooth manifold, and $W \subset M^n$ is open, then $W$ inherits a smooth atlas from $M^n$: if $\varphi\colon U \to V$ is a chart for $M^n$, then $\varphi|_{U\cap W} \colon U \cap W \to \varphi(U \cap W)$ is a chart for $W$.

**Example 2.** Consider the $n$-sphere:

$$S^n = \left\{ (x_1, \ldots, x_{n+1}) \in \mathbf{R}^{n+1} \;\middle|\; \sum_i x_i^2 = 1 \right\}.$$

We claim $S^n$ is a smooth manifold. To give an atlas, set

$$U_{x_i>0} = \{(x_1, \ldots, x_{n+1}) \in S^n \mid x_i > 0\}$$

$$U_{x_i<0} = \{(x_1, \ldots, x_{n+1}) \in S^n \mid x_i < 0\}$$

for all $1 \leq i \leq n+1$, and define charts $\varphi_{x_i>0}\colon U_{x_i>0} \to V_{x_i>0}$ by

$$\varphi(x_1, \ldots, x_{n+1}) = (x_1, \ldots, \hat{x}_i, \ldots, x_{n+1})$$

and similarly define charts $\varphi_{x_i<0}\colon U_{x_i<0} \to V_{x_i<0}$. These clearly cover $S^n$. We must show the transition functions are smooth. For concreteness we show smoothes for $\tau_{12}\colon \varphi_{x_1>0}(U_{x_1>0} \cap U_{x_2>0}) \to \varphi_{x_2>0}(U_{x_1>0} \cap U_{x_2>0})$; the rest of the transition functions are similar. Let

$$\Lambda = U_{x_1>0} \cap U_{x_2>0} = \{(x_1, \ldots, x_{n+1}) \in S^n \mid x_1 > 0, x_2 > 0\}.$$

[1] Sometimes, one defines a smooth manifold to be a manifold equipped with a maximal smooth atlas, but this requires Zorn's Lemma, and is perhaps less elegant than our approach

Then $\varphi^{-1}_{x_1>0}(y_1,\ldots,y_n) = (\sqrt{1-y_1^2-\ldots-y_n^2}, y_1,\ldots,y_n)$, and $\varphi_{x_2>0}(\sqrt{1-y_1^2-\ldots-y_n^2}, y_1,\ldots,y_n) = (\sqrt{1-y_1^2-\ldots-y_n^2}, y_2,\ldots,y_n)$, so

$$\tau_{12}(y_1,\ldots,y_n) = (\sqrt{1-y_1^2-\ldots-y_n^2}, y_2,\ldots,y_n)$$

which is smooth.

**Example 3.** Consider the $n$-dimensional real projective space

$$\mathbf{R}P^n = S^n/\sim$$

where $\sim$ identifies pairs of antipodal points on $S^n$, i.e. points $x, y \in S^n \subseteq \mathbf{R}^{n+1}$ with $y = -x$. To see this is a manifold, just use the charts $\varphi_{x_i>0}\colon \tilde{U}_{x_i>0} \to V_{x_i>0}$, where here $\tilde{U}_{x_i>0}$ is the open set of $\mathbf{R}P^n$ whose pre image under the natural projection from $S^n$ is $U_{x_i>0}$. Note that $\mathbf{R}P^2$ cannot be obviously embedded into $\mathbf{R}^3$, and in fact, it can't be embedded into $\mathbf{R}^3$ at all. There exists, however, an embedding of $\mathbf{R}P^2$ into $\mathbf{R}^4$.

**Example 4.** Let $M_1^n$ and $M_2^n$ be smooth manifolds. Then $M_1^{n_1} \times M_2^{n_2}$ is a smooth manifold. Just take products of charts. The ***n*-torus** $T^n$ is a key example of a manifold we construct with the product:

$$T^n := \underbrace{S^1 \times \ldots \times S^1}_{n \text{ times}}$$

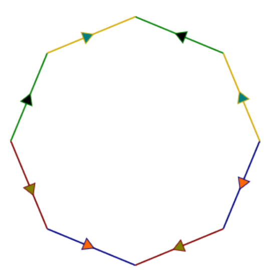

Figure 1: Fundamental polygon of genus 2 surface. Source: http://www.math.cornell.edu/ mec/Winter2009/Victor/part4(2).png

**Example 5.** Consider the space in Figure 1 which is the quotient of the octagon in the plane formed by identifying the sides with matching colors with each other with the prescribed orientations:

One should go through the process of convincing oneself that this space is a 2-holed donut, after making the appropriate identifications. To give a smooth atlas, we identify three kinds of charts.

(1) $U$ is an open subset in the interior of the octagon, the chart is the identity map $\mathrm{id}\colon U \to U$.

(2) The charts on discs formed by two half discs along interiors of identified edges.

(3) The union of open sectors around vertices, with the chart properly squashing each sector to fit together into a circle.

### 1.2 Smooth functions on manifolds

We are now ready to define smooth functions on manifolds.

**Definition 5.** Let $M^n$ be a smooth manifold with $W \subseteq M^n$ an open subset, and let $f\colon W^{\to}\mathbf{R}$ be a function. We say that $f$ is **smooth** if, for all charts $\varphi\colon U \to V \subseteq \mathbf{R}^n$, the composition $f \circ \varphi^{-1}\colon \varphi(W) \to \mathbf{R}$ is smooth.

## 2 8/27/2015: The Tangent Bundle and Smooth Maps

### 2.1 More on smooth functions on manifolds

We finished with a quick definition of "smooth function" on a manifold last time. Let's review that.

**Definition 6.** Let $M$ be a smooth manifold, $f\colon M^n \to \mathbf{R}$ is a function. We say that $f$ is **smooth at a point** $p \in M^n$ if, for a chart $\varphi\colon U \to V$, $U \subset M^n, V \subset \mathbf{R}^n$ with $p \in U$, the function $g\colon V \to \mathbf{R}$ given by $g = f \circ^{-1} \varphi$ is smooth at $\varphi(p)$. We say that $f$ is smooth if $f$ is smooth at all points.

**Remark 1.** This is well-defined since transition functions are smooth: If $\varphi_1\colon U_1 \to V_1$ is another chart with $p \in U$, then on $\varphi_1(U \cap U_1)$ we can factor $f \circ \varphi^{-1}$ as

$$\varphi_1(U \cap U_1) \xrightarrow{\varphi_1^{-1}} U \cap U_1 \xrightarrow{\varphi} \varphi(U \cap U_1) \xrightarrow{\varphi^{-1}} U \cap U_1 \xrightarrow{f} \mathbf{R}.$$

**Alternate point of view:** Let $\{\varphi_1\colon U_i \to V_i\}_{i\in I}$ be an atlas for $M$. One can write

$$M^n = \bigsqcup_{i\in I} V_i / \sim,$$

where $\sim$ identifies $\varphi_i(p)$ and $\varphi_j(p)$ for all $p \in U_i \cap U_j$ and $i \in j$ [2]. Then a smooth function $f\colon M^n \to \mathbf{R}$ is the same as a collection of smooth functions $f\colon V_i \to \mathbf{R}$ which agree on the overlaps.

[2] In fancy language, we have written $M^n$ as a colimit of its atlas

For example, if $\tau_{ij}\colon \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ is a transition function, then

$$f_i|_{\varphi_i(U_i \cap U_j)} = f_j|_{\varphi_j(U_i \cap U_j)} \circ \tau_{ij}.$$

## 2.2 The Tangent Bundle

Recall the situation in Euclidean space. Let $V \subseteq \mathbf{R}^n$ be open. The **tangent space** of $V$ at a point $p \in V$ is then just the vector space $\mathbf{R}^n$. We write this as $T_pV$. The **tangent bundle** of $V$ is $TV = V \times \mathbf{R}^n$.

Given a smooth function on $V_1, V_2$ open subsets of $\mathbf{R}^n$, $\psi\colon V_1 \to V_2$, then for all points $p \in V_1$ we get the derivative

$$D_p\psi\colon T_pV_1 \to T_{\psi(p)}V_2.$$

In coordinates, $D_p\psi$ is the linear map whose matrix is the matrix of partial derivatives,

$$D_p\psi = \left(\frac{\partial \psi_i}{\partial x_j}\right).$$

If $\psi_1\colon V_1 \to V_2$ and $\psi_2\colon V_2 \to V_3$ are smooth maps, the **chain rule** says that for all $p \in V_1$, we have

$$D_p(\psi_2 \circ \psi_1) = [D_{\psi_1(p)}\psi_1] \circ [D_p\psi_1]$$

Our eventual goal is to globalize these notions to smooth manifolds. For a smooth function $f\colon V_1 \to V_2$, the $D_p\psi$ piece together to give a function

$$Df\colon TV_1 \to TV_2.$$

For $\psi_1\colon V_1 \to V_2$ and $\psi_2\colon V_2 \to V_3$, the chain rule simply becomes

$$D(\psi_2 \circ \psi_1) = D\psi_2 \circ D\psi_1.$$

Let's return to manifolds. $M^n$ is a smooth manifold with atlas $\mathcal{A} = \{\varphi_i\colon U_i \to V_i\}_{i \in I}$. Recall from above that

$$M^n = \bigsqcup_{i \in I} / \sim$$

where $\sim$ comes from the transition functions $\tau_{ij}$. The **tangent bundle** of $M^n$ is

$$TM = \bigsqcup_{i \in I} TV_i / \sim$$

where $\sim$ comes from the derivatives of transition functions $D\tau_{ij}$.[3]

For each $p \in M^n$, we have a tangent space

$$T_pM^n = \bigsqcup_{\substack{i \in I \\ p \in U_i}} T_{\varphi_i(p)}V_i \big/ \sim .$$

On the homework, you will check that this means that $T_pM^n$ is an $\mathbf{R}$-vector space of $\dim n$. A choice of chart containing $p$ gives a basis.

### 2.3 Directional Derivatives

Given $v \in T_pM^n$, and a smooth function $f\colon M^n \to \mathbf{R}$, the **directional derivative** of $f$ in the direction of $v$ is the following *number*:

- Choose a chart $\varphi\colon U_1 \to V_1 \subset \mathbf{R}^n$ with $p \in U_1$.
- $v$ is identified with $v_1 \in T_{\varphi(p)}V_1$.
- Take directional derivative of $f \circ \varphi^{-1} \to \mathbf{R}$ in the direction of $v$.

This number is actually well-defined: If $\varphi_2\colon U_2 \to V_2$ is another choice with $p \in U_2$, then $v \in T_pM$ is identified with $v_2 \in T_{\varphi_2(p)}V_2$. We have by definition that

$$v_2 = [D_{\varphi_2(p)}\tau_{12}](v_1)$$

$$f \circ \varphi_2^{-1}|_{\varphi_2(U_1 \cap U_2)} = f \circ \tau_{12} \circ \varphi_1^{-1}|_{\varphi_1(U_1 \cap U_2)}.$$

The chain rule for directional derivatives implies that the directional derivative of $f \circ \varphi_2^{-1}$ in direction $v_2$ is the same as that of $f \circ \varphi_1^{-1}$ in direction $v_1$.

### 2.4 Smooth maps and embeddings into $\mathbf{R}^m$

Let $M^n$ be a smooth manifold. Then a smooth map $f\colon M^n \to \mathbf{R}^m$ is one whose coordinate functions $f_i\colon M^n \to \mathbf{R}$ are smooth for $1 \le i \le m$.

Given such a map, we get for each $p \in M^n$ a linear map

$$D_pf\colon T_pM^n \to T_{f(p)}\mathbf{R}^m.$$

This map works by simply choosing a chart $\varphi\colon U \to V$ with $p \in U$, then you get a smooth map $f \circ \varphi^{-1} \to \mathbf{R}^m$, then take the derivative and use the identification of $T_pM^n$ with $T_{\varphi(p)}V$.

[3]For those who know about fiber bundles, the chain rule for derivatives ensures the cocycle condition holds.

**Definition 7.** We say that a smooth $f\colon M^n \to \mathbf{R}^m$ is an **embedding** if

- $f$ is homeomorphic onto its image (a topological embedding), and
- Each $Df_p\colon T_pM^n \to T_p\mathbf{R}^m$ is injective.

Given such an embedding, one can identify $TM$ with

$$\{(f(p), [D_pf](v)) \in T\mathbf{R}^n (= \mathbf{R}^m \times \mathbf{R}^m) \parallel p \in M^n, v \in T_pM^n\}$$

For example, $S^2$ comes with a natural embedding into $\mathbf{R}^3$, namely the natural injection $S^2 \hookrightarrow \mathbf{R}^3$. Similarly, $S^n \hookrightarrow \mathbf{R}^{n+1}$.

We now show every compact manifold embeds into $\mathbf{R}^n$:

**Theorem 1.** *Let $M^n$ is a compact $n$-manifold. Then for some $m >> 0$, there exists an embedding $f\colon M^n \to \mathbf{R}^m$.*

**Remark:** Whitney showed that one can take $m = 2n$. Later we will show that one can take $m = 2n + 1$.

*Proof.* Since $M^n$ is compact, $M^n$ has a finite atlas

$$\mathcal{A} = \{\varphi_i\colon U_i \to V_i\}_{i=1}^{\ell}.$$

We can also find open subsets $W_i \subset U_i$ such that the $W_i$ also cover $M^n$ and the closure of $W_i$ in $U_i$ is compact. We can now find smooth functions

$$\psi_i\colon V_i \to \mathbf{R}^n$$

such that

(a) $\psi_i|_{\varphi_i(w_i)} = \text{id}$, and

(b) $\psi_i$ has compact suport, i.e. $\overline{\{x \in V_i \mid \psi_i(x) \neq 0\}}$ is compact.

By item (b), we can define $\eta_i\colon M^n \to \mathbf{R}^n$ such that

$$\eta_i|_{U_i} = \psi_i \circ \varphi_i$$

$$\eta_i|_{M^n \setminus V_i} = 0$$

and this is a smooth map. Define

$$f\colon M^n \to \mathbf{R}^{\ell n}$$

by

$$f(p) = (\eta_1(p), \eta_2(p), \dots, \eta_\ell(p)).$$

This $f$ is an embedding. It's clear it's a topological embedding, and on the homework you will check that is in injective on the tangent spaces. □

# 3 9/1/2015: More smooth maps, regular values

## 3.1 Finally, we define a smooth map of two manifolds

There are some maps that really "ought" to be smooth. For example, the embedding $i \hookrightarrow \mathbf{R}^{n+1}$ should be smooth. But note that $i^{-1}(\mathbf{R}^n)$ is not contained in a single chart. Similarly, note if we take the projection $\pi\colon \mathbf{R} \to S^1$ given by $t \mapsto (\cos t, \sin t)$, which again ought to be smooth, $\pi(\mathbf{R})$ is not contained in a single chart. The point: *smooth maps do not have to take charts to charts.*

This could cause some trouble for any definition of smooth, depending on the atlas we choose. So from henceforth, we make the following convention: all of our atlases will be maximal (which you can do because of Zorn's lemma). The key property is: if $\varphi\colon U \to V$ is a chart and $U' \subseteq U$ is open, then $\varphi_{U'}\colon U' \to \varphi(U')$ is also a chart.

This allows us to make the following definition

**Definition 8.** A function $f\colon M_1^{n_1} \to M_2^{n_2}$ is **smooth at** $p \in M_1^{n_1}$ if there exists charts $\varphi_1\colon U_1 \to V_1$ for $M_1^{n_1}$ with $p \in U_1$ and $\varphi_2\colon U_2 \to V_2$ for $M_2^{n_2}$ with $f(p) \in U_2$ such that $f(U_1) \subseteq U_2$ and the composition

$$\mathbf{R}^{n_2} \supseteq V_1 \xrightarrow{\varphi_1^{-1}} U_1 \xrightarrow{f} U_2 \xrightarrow{\varphi_2} V_2 \subseteq \mathbf{R}^{n_2}$$

is smooth at $\varphi_1(p)$. We say $f$ **is smooth** if $f$ is smooth at all $p \in M_1^{n_1}$.

A smooth map $f\colon M_1^{n_1} \to M_2^{n_2}$ induces a map $Df\colon TM_1^{n_1} \to TM_2^{n_2}$ is the obvious way, namely using local charts. By the chain rule in each chart, it follows that for a composition of smooth maps

$$M_1^{n_1} \xrightarrow{f} M_2^{n_2} \xrightarrow{g} M_3^{n_3}$$

then

$$D(g \circ f) = Dg \circ Df\colon TM_1^{n_1} \to TM_3^{n_3}$$

## 3.2 Local structure of manifolds

We need to talk about what smooth maps look like locally.

**Lemma 1.** *Let $f\colon M_1^{n_1} \to M_2^{n_2}$ be smooth, $p \in M_1 n_1$. If $D_p f\colon T_p M_1^{n_1} \to T_{f(p)} M_2 n_2$ is an isomorphism, then $f$ is a local diffeomorphism at $p$; i.e., there exists a neighborhood $U$ of $p$ such that $f(U)$ is an open subset of $M_2^{n_2}$ and $f|_U\colon U \to f(U)$ is a diffeomorphism.*

*Proof.* Without loss of generality, one can assume that $M_1^{n_1} \subseteq \mathbf{R}^{n_1}$ is open and $M_2^{n_2} \subseteq \mathbf{R}^{n_2}$ is open (just replace with open neighborhood of $p$, $f(p)$). Then this is just the statement of the inverse function theorem. □

**Lemma 2.** *Let $f\colon M_1^{n_1} \to M_2^{n_2}$ be smooth, $p \in M_1^{n_1}$. If $D_p f\colon T_p M_1^{n_1} \to T_{f(p)} M_2^{n_2}$ is injective, then one can choose local coordinates around $p$ and $f(p)$ via some charts $\varphi_1\colon U_1 \to V_1$ with $p \in U_1$, $\varphi_2\colon U_2 \to V_2$ with $f(p) \in U_2$ such that in those local coordinates,*

$$f \circ \varphi_1^{-1}\colon V_1 \hookrightarrow V_1 \times \mathbf{R}^{n_2 - n_1} \subseteq V_2.$$

*i.e., $f \circ \varphi^{-1}$ is the natural injection.*

*Proof.* Without loss of generality, $M_1^{n_1} \subseteq \mathbf{R}^{n_1}$ and $M_2 \subseteq \mathbf{R}^{n_2}$ open. Also, composing the inclusion $M_2 \subseteq \mathbf{R}^{n_2}$ with a linear diffeomorphism, one can assume that

$$D_p f\colon T_p M_1^{n_1} \to T_{f(p)} M_2^{n_2}$$

is the usual injection $\mathbf{R}^{n_1} \hookrightarrow \mathbf{R}^{n_2}$. Then we have the map

$$M_1^{n_1} \times \mathbf{R}^{n_2 - n_1} \xrightarrow{F} \mathbf{R}^{n_2}$$

given by $F(m, v) = f(m) + v$. By our work above, the derivative of $F$ at $(p, 0)$ is the identity map. By Lemma 2, $F$ is a local diffeomorphism at $(p, 0)$. The function $f$ is just the composition

$$M_1^{n_1} \hookrightarrow M_1^{n_1} \times \mathbf{R}^{n_2 - n_1} \xrightarrow{F} \mathbf{R}^{n_2}.$$

Since $F$ is a local diffeomorphism at $(p, 0)$, one can find an open subset $U$ of $(p, 0)$ such that $F|U$ is a diffeomorphism shrinking $U$, and one can also assume that $F(U) \subseteq M_2^{n_2}$. Then replace the chart we have on $M_2$ with a smaller chart

$$M_2^{n_2} \supseteq F(U) \xrightarrow{F^{-1}} U \subseteq M^{n_1} \times \mathbf{R}^{n_2 - n_1}.$$

Using this chart, $f$ has the desired form.

□

### 3.3 Regular values

**Definition 9.** Let $f\colon M_1^{n_1} \to M_2^{n_2}$ be smooth. A point $p \in M_1^{n_1}$ is a **regular point** if $D_p f$ is surjective. A point $q \in M_2^{n_2}$ is a **regular value** if all points in $f^{-1}(q)$ are regular points.

**Theorem 2.** *If $f\colon M_1^{n_1} \to M_2^{n_2}$ is smooth, $q \in M_2^{n_2}$ is a regular value, then $f^{-1}(q)$ is an embedded submanifold of $M_1^{n_1}$ of dimension $n_1 - n_2$.*

**Example 6.** If $n_2 > n_1$, then no point of $M_1^{n_1}$ can be a regular point. Hence if $q \in M_2^{n_2}$ is a regular value, then $f^{-1}(q) = \emptyset$.

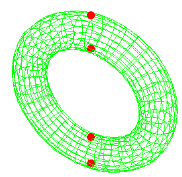

Figure 2: Critical points of the torus height function. Source: http://i.stack.imgur.com/refrl.gif

**Example 7.** Consider the 2-torus $T$ embedded in $\mathbf{R}^3$ as in Figure 2, with the red points, from bottom to top, having coordinates $(0,0,0), (0,0,1), (0,0,2), (0,0,3)$. The height function $h\colon T \to \mathbf{R}$ is given by $h(x,y,z) = z$. The map is clearly smooth, and the critical points are given by the four red points. The regular values are the real numbers in the set $\mathbf{R}\backslash\{0,1,2,3\}$. The pre image of a regular value $t \in (0,1)$ is diffeomorphic to $S^1$. The pre image of a regular value $t \in (1,2)$ is diffeomorphic to $S_1 \sqcup S_1$. The pre image of a regular value $t \in (2,3)$ is again diffeomorphic to $S^1$.

**Example 8.** Consider the 2-sphere with three disjoint copies of $S^1$ tracing out three distinct circles on $S^2$. Collapse the region of the sphere bounded by all three of these embedded circles to a single point. This quotient is the wedge of three 2-spheres, $S^2 \vee S^2 \vee S^2$. Then one can identify three the three $S^2$'s to one $S^2$. Let $f$ be this map $S^2 \to S^2$. If one is careful, one can arrange that $f$ is smooth. The critical points are those in the interior of the region bounded by the three circles, together with the points on the circles themselves (derivative is 0 there, hence not surjective). Regular values are all points but the south pole. If $t \in S^2$, then $f^{-1}(t)$ is three points, which is a manifold of dimension 0 [4]

[4] One day, I will add pictures to this example. Then again, I may never get around to that.

# 4 Immersions, submersions, and the Fundamental Theorem of Algebra

## 4.1 Immersions and submersions

The professor would like to make sure everyone knows this word:

**Definition 10.** A smooth map $f\colon M_1 \to M_2$ is an **immersion** at $p$ if $D_p f\colon T_p M_1 \to T_{f(p)} M_2$ is injective.

**Example 9.** This $\mathbf{R} \to \mathbf{R}^2$ is an immersion:

Here's a better version of Lemma 2 from last time, with a clearer proof (but really its the same).

**Theorem 3. (Local Immersion Theorem)** *Let $f\colon M_1^{n_1} \to M_2^{n_2}$ be a smooth immersion at $p \in M_1$. Then there exists an open neighborhood $U_1 \subseteq M_1$ of $p$ and $U_2 \subseteq M_2$ with $f(U) \subseteq U_2$ together with an open set $W \subseteq \mathbf{R}^{n_1 - n_2}$ and a point $w_0 \in W$ and a diffeomorphism $\psi\colon U_2 \to U_1 \times W$ such that the composition*

$$U_1 \xrightarrow{f} U_2 \xrightarrow{\psi} U_1 \times W$$

*takes $u \in U_1$ to $(u, w_0)\imath U_1 \times W$.*

*Proof.* Choose charts $\varphi_1\colon U_1 \to V_1 \subseteq \mathbf{R}^{n_1}$, $\varphi_2\colon U_2 \to V_2 \subseteq \mathbf{R}^{n_2}$ such that $p \in U_1$ and $f(U_1) \subseteq U_2$. Let $F\colon V_1 \to V_2$ be an expression for $f$ in local coordinates:

$$V_1 \xrightarrow{\varphi_1^{-1}} U_1 \xrightarrow{f} U_2 \xrightarrow{\varphi_2} V_2$$

i.e. $F = \varphi_2 \circ f \circ \varphi_1^{-1}$. Set $q = \varphi_2(p)$. Then $F$ is an immersion at $q$, and it suffices to prove the theorem for $F$.

By assumption, $D_q F\colon T_q V_1 \to T_{F(q)} V_2$ is injective. Choose a vector subspace $X \subseteq T_{F(q)} V_2$ such that $T_{f(p)} V_2 = \mathrm{Im}(D_q F) \oplus X$. Then $X \cong \mathbf{R}^{n_1 - n_2}$. Note that $T_{(q,0)}(V_1 \times X) = T_q V_1 \oplus T_0 X$. Define

$$G\colon V_1 \times X \to \mathbf{R}^{n_2}$$

$$(v, x) \mapsto F(v) + x.$$

By construction, $D_{(q,0)} G\colon T_{(q,0)}(V_1 \times X) \to T_{F(q)} \mathbf{R}^{n_2}$ is an isomorphism, using the direct sum decomposition above. Hence, the inverse function theorem says that $G$ is a local diffeomorphism at $(q, 0)$. Therefore, we can find open subsets $V_1' \times W \subseteq V_1 \times X$ and $V_2' subseteq V_2$ such that

$(a,b) \in V_1' \times W$ and $G(V_1' \times W) \subseteq V_2'$, and such that $G$ restricts to a diffeomorphism from $V_1' \times W \to V_2'$. Therefore, the composition $H = G^{-1} \circ F$ takes $v \in V_1'$ to $(v,0) \in V_1' \times W$.

□

There is a similar theorem for submersions.

**Definition 11.** A smooth map $f\colon M_1 \to M_2$ is an **submersion** at $p$ if $D_p f\colon T_p M_1 \to T_{f(p)} M_2$ is surjective.

**Theorem 4. (Local Submersion Theorem)** *Let $f\colon M_1^{n_1} \to M_2^{n_2}$ be a smooth submersion at $p \in M_1$. Then there exists an open neighborhood $U_1 \subseteq M_1$ of $p$ and $U_2 \subseteq M_2$ with $f(U) \subseteq U_2$ together with an open set $W \subseteq \mathbf{R}^{n_1-n_2}$ and a diffeomorphism $\psi\colon U_2 \times W \to U_1$ such that the composition*

$$U_2 \times W \xrightarrow{\psi} U_1 \xrightarrow{f} U_2$$

*takes $(u,w) \in U_2 \times W$ to $u \in U_2$.*

*Proof.* The proof is isomorphic to that of the local immersion theorem, so we omit. □

## 4.2 Regular values and submanifolds, and Sard's Theorem

We now have a theorem that basically will pop out of the local submersion theorem.

**Theorem 5.** *Let $f\colon M_1^{n_1} \to M_2^{n_2}$ be smooth, $q \in M_2$ a regular value. Then $f^{-1}(q)$ is a smooth $(n_1 - n_2)$-dimensional manifold embedded in $M_1$, and for $p \in f^{-1}(q)$, $T_p f^{-1}(q) = \ker(D_p f\colon T_p M_1 \to T_{f(p)} M_2)$.*

*Proof.* Let $p \in f^{-1}(q)$. The local submersion theorem implies that there exists $U_1 \subseteq M - 1$ of $p$ and $U_2 \subseteq M_2$ such that $f(U_1) \subseteq U_2$ and $W \subseteq \mathbf{R}^{n_1-n_2}$, and a diffeomorphism $\psi\colon U_2 \times W \to U_1$ such that the composition

$$U_2 \times W \xrightarrow{\psi} U_1 \xrightarrow{f} U_2$$

takes $(u,w) \in U_2 \times W$ to $u \in U_2$. Then $\psi^{-1}$ restricts to a diffeomorphism from $f^{-1}(q) \cap U_1$ to $\{q\} \times W$, i.e. $p \in f^{-1}(q)$ has a neighborhood diffeomorphic to $W \subseteq \mathbf{R}^{n_1-n_2}$.

□

The following theorem is essential to differential topology, because it tells us most points are regular values. The proof is mostly analytic, and is not really that useful in other parts of topology.

**Theorem 6. Sard's Theorem** *Let $f\colon M_1 \to M_2$ be smooth. Then the critical points of $M_2$ form a set of measure zero in $M_2$.*

**Example 10.** Let $f\colon \mathbf{R}^{n+1} \to \mathbf{R}$ be the map $f(x_1, \ldots, x_{n+1}) = x_1^2 + \ldots + x_{n+1}^2$. The derivative

$$D_p f\colon T_p\mathbf{R}^{n+1} \to T_{f(p)}\mathbf{R}$$

is given by the matrix

$$\begin{pmatrix} 2p_1 & 2p_2 & \ldots & 2p_{n+1} \end{pmatrix}$$

This is surjective if and only if $p \neq 0$. Therefore, all nonzero points of $\mathbf{R}$ are regular values, and in particular, $S^n = f^{-1}(1)$ is an $n+1-1 = n$-dimensional manifolds embedded in $\mathbf{R}^{n+1}$.

**Example 11.** We can identify the set $\mathrm{Mat}_n(\mathbf{R})$ of $n \times n$ real matrices with the space $\mathbf{R}^{n^2}$ with the standard Euclidean topology. Define

$$f\colon \mathrm{Mat}_n \to \mathbf{R}$$

$$f(A) = \det A.$$

We claim that $f$ is a submersion at all point $A \in \mathrm{Mat}_n(\mathbf{R})$ such that $\det(A) \neq 0$. From this claim, it will follow that the nonzero reals are regular values of $f$, so $SL_2(\mathbf{R}) = f^{-1}(1)$ is a smooth manifolds of dimension $n^2 - 1$.

To prove the claim, consider $A \in \mathrm{Mat}_n(\mathbf{R})$ with $\det(A) \neq 0$. Then define $g\colon \mathbf{R} \to \mathrm{Mat}_n(\mathbf{R})$ by $g(t) = tA$. Then

$$(f \circ g)(t) = \det(tA) = t^n \det A.$$

Thus,

$$D_1(f \circ g)\colon T_1\mathbf{R} \to T_{\det A}\mathbf{R}$$

is multiplication by $n\det(A) \neq 0$. Thus, $D_1(f \circ g)$ is surjective. The chain rule implies that $D_A f$ is surjective. The claim is proved.

## 4.3 The Fundamental Theorem of Algebra

Warning: the following proof is so beautiful, we may stay past the end of class to finish the proof. We start with a lemma.

**Lemma 3.** *Let $f\colon M^n \to M^n$ be smooth. Let $U \subseteq M^n$ be the set of regular values. Assume that $M^n$ is compact and has finitely many non-regular values. Then the function*

$$g\colon U \to \mathbf{Z}_{\geq 0}$$

$$t \mapsto |f^{-1}(t)|$$

*is constant.*

*Proof.* $U$ is connected (compact minus finitely many points), so it suffices to show that $g$ is locally constant. Consider $q \in U$, and write $f^{-1}(q) = \{p_1, \ldots, p_k\}$. We know that $f$ is a local diffeomorphism at each $p_i$. Therefore, there exists neighborhoods $U_i$ containing $p_i$, which we may take to be disjoint after shrinking each one, and neighborhoods $W_i$ of $q$ such that $f|_{U_i}$ is a diffeomorphism onto $W_i$. Set

$$W = (\bigcap_{i=1}^{k} W_i) \backslash f(M \backslash (\bigcup_{i=1}^{k} U_i'))$$

and note that $q \in W$, so $W$ is a nonempty, open set. We know that $f|_{U_i'}$ is a diffeomorphism onto $W$. To show that $g$ is locally constant, it is enough to show that $f^{-1}(W) = U_1' \cup \cdots \cup U_k'$. Clearly $U_1' \cup \cdots \cup U_k' \subseteq f^{-1}(W)$. For the reverse inclusion, take $q' \in f^{-1}(W)$. Then $f(q') \in W$. Since $W$ only contains points that are the images of points in $U_1' \ldots, U_k'$, we must have $q' \in U_1' \cup \ldots \cup U_k'$. □

**Theorem 7. (Fundamental Theorem of Algebra)** *If $f(z)$ is a nonconstant $\mathbf{C}$-polynomial, then $f(z)$ has a root.*

*Proof.* Use the stereographic projection of $S^2$: the charts are

$$\psi_1\colon U_1 = S^2 \backslash (0,0,1) \to \mathbf{R}^2.$$

$$\psi_2\colon U_2 = S^2 \backslash (0,0,-1) \to \mathbf{R}^2.$$

$$\psi_1(p) = \text{ intersection of } \mathbf{R}^2 \text{ with lines through } (0,0,1) \text{ and } p$$

$$\psi_2(p) = \text{ intersection of } \mathbf{R}^2 \text{ with lines through } (0,0,-1) \text{ and } p$$

This gives an alternate atlas for $S^2$. On the homework, you will show that this atlas is compatible with the usual one. Note that if we view $\mathbf{R}^2$ as the complex plane, this covers the sphere minus a point with a copy of the complex plane, and we have two of these complex plane covering the sphere, one for each pole we omit from the sphere.

Now, define $F\colon S^2 \to S^2$ as follows:

$$F(p) = p \text{ if } p = (0,0,1)$$

$$F(p) = \varphi_1(f(\varphi_1^{-1}(p))) \text{ if } p \neq (0,0,1).$$

This is a smooth map, and $F(U_1) \subseteq U_1$. The expression for $F$ with respect to local coordinates $\varphi_1\colon U_1 \to \mathbf{C}$ is simply $f(z)$. The derivative at $z_0 \in \mathbf{C}$ be surjective unless $f'(z_0) = 0$, which is only true for finitely many zeros. So $F$ has only finitely many non regular values. Let $U \subseteq S^2$ be the set of regular values. We know that $p \in U$ implies $|F^{-1}(p)| \in \mathbf{Z}$ is constant. We certainly hit some point of $U$, so $F^{-1}(p) \neq \emptyset$ for any $p \in U$. Clearly, $F^{-1}(p) \neq \emptyset$ for $p \in S^2 \backslash U$, so $F$ is surjective, and hence $f$ has a zero. □

# 5 9/8/2015: Manifold with boundary and the Brouwer Fixed Point Theorem

Many spaces are "almost manifolds".

**Example 12.** The interval $[0,1]$ is not a manifold at 0, 1, but is everywhere else.

**Example 13.** The closed unit disc $\mathbf{D}^n = \{x \in \mathbf{R}^n \mid ||x|| \leq 1\}$ is not a manifold on it boundary $S^{n-1} \subseteq \mathbf{D}^n$.

Both of these examples are examples of *manifolds with boundary.* First, we need to know what it means to be smooth on a non-open subset of $\mathbf{R}^n$.

**Definition 12.** Let $X \subseteq \mathbf{R}^n$ be any subset. A function $f\colon X \to \mathbf{R}^m$ is **smooth** if there exists an open set $U \subseteq \mathbf{R}^n$ with $X \subseteq U$ and a smooth function $g\colon U \to \mathbf{R}^m$ such that $g|_X = f$.

Let's introduce some notation

$$\mathbf{H}^n = \{(x_1, \ldots, x_n) \in \mathbf{R}^n \mid x_n \geq 0\}$$

Note that

$$\partial \mathbf{H}^n = \{(x_1, \ldots, x_n) \mid x_n = 0\}$$

(we read "$\partial$" as "boundary").

**Definition 13.** A **smooth manifold with boundary** is a paracompact, Hausdorff space $M^n$ equipped with a smooth atlas $\{\varphi_i \colon U_i \to V_i\}$ defined almost exactly as before, but now $V_i$ is an open subset of $\mathbf{H}^n$, where $\mathbf{H}^n$ is given the subspace topology from $\mathbf{R}^n$.

The tangent bundle $TM^n$ is defined exactly as before.

For every $p \in M^n$, there are two possibilities:

(a) there exists a neighborhood $U \subseteq M^n$ of $p$ homeomorphic to an open subset of $\mathbf{R}^n$.

(b) there exists a neighborhood $U \subseteq M^n$ of $p$ homeomorphic to an open subset $V \subseteq \mathbf{H}^n$, but not open in $\mathbf{R}^n$. Then there is $V \cap \partial\mathbf{H}^n \neq \emptyset$ and $p$ is identified with a point of $\partial H^n$.

One needs the technique of local homology in order to formally prove this, but we will accept it as intuitively true.

An important clarification: for $U \subseteq \mathbf{H}^n$ open, define $TU = U \times \mathbf{R}^n$. If $p \in \partial\mathbf{H}^n \cap U$, we still have $T_pU = \mathbf{R}^n$. Tangent vectors can "point outwards".

One of the most useful tools for proving a space is a manifold is to show it arises as the pullback of a regular value.

**Theorem 8.** *Let $M^n$ be a smooth n-manifold, $f \colon M^n \mathbf{R}$ be smooth. Then*

*(a) if $a \in \mathbf{R}$ is a regular value, then $f^{-1}((-\infty, a])$ and $f^{-1}([a, \infty))$ are smooth n-manifolds with boundary $f^{-1}(a)$.*

*(b) If $a, b \in \mathbf{R}$ are regular values with $a < b$, then $f^{-1}([a, b])$ is a smooth n-manifold with boundary $f^{-1}(a) \cup f^{-1}(b)$.*

*Proof.* The same as for smooth manifolds, using the local submersion theorem. □

**Example 14.** $f \colon \mathbf{R}^n \to \mathbf{R}$ given by $f(x_1, \ldots, x_n) = x_1^2 + \ldots + x_n^2$. Then 1 is a regular value, and hence $\mathbf{D}^n = f^{-1}((-\infty, 1])$ is a smooth $n$-manifold with boundary $f^{-1}(1) = S^{n-1} \subseteq \mathbf{D}^n$.

**Example 15.** Consider the 2-torus $T^2$, a the smooth height function $f \colon T^2 \to \mathbf{R}$ from Example . Pick two regular values $t_1, t_2$, then the pullback is a manifold with boundary.

Here's a theorem, whose proof is not really instructive, and is in Milnor's book.

**Theorem 9.** *Let $M$ be a compact, connected 1-manifold with boundary. Then either $M \cong [0, 1]$ or $M \cong S^1$.*

Here's a generalization of Theorem 8, whose proof is also in Milnor's book.

**Theorem 10.** *Let $f\colon M_1^{n_1} \to M_2^{n_2}$ be a smooth map between smooth manifolds with boundary. Assume that $p \in M_n^{n_2}$ is a regular value for both $f$ and $f\big|_{\partial M_1^{n_1}}$. Then $f^{-1}(p)$ is a smooth $(n_1 - n_2)$-dimensional manifold with boundary, and $\partial f^{-1}(p) = (f\big|_{\partial M_1^{n_1}})^{-1}(p) \subseteq \partial M_1^{n_1}$*

**Theorem 11. (The Brouwer Fixed Point Theorem)** *Let $f\colon \mathbf{D}^n \to \mathbf{D}^n$ be a continuous map. Then there exists $p \in \mathbf{D}^n$ such that $f(p) = p$.*

Note that this is obviously true for $n = 1$. Just use the intermediate value theorem to show that the lines $y = x$ and $y = f(x)$ intersect.

The key ingredient is the following lemma:

**Lemma 4.** *There does not exist a smooth map $g\colon \mathbf{D}^n \to \partial\mathbf{D}^n$ such that $g|_{\partial\mathbf{D}^n} = \mathrm{id}$.*

*Proof.* Assume such a $g$ exists. Let $q \in \partial\mathbf{D}^n$ be a regular value for $g$ (one exists by Sard's Theorem). Sicne $g|_{\partial\mathbf{D}^n} = \mathrm{id}$, $q$ is also a regular value for $g|_{\partial\mathbf{D}^n}$. Therefore, $g^{-1}(q)$ is a smooth $(n-(n-1)) = 1$-manifold with boundary, and $\partial g^{-1}(q) = (g|_{\partial\mathbf{D}^n})^{-1}(q) = \{q\}$. But any compact 1-manifold has an even number of boundary points, so this is a contradiction! □

*Proof.* (Theorem **??**) Suppose $f\colon \mathbf{D}^n \to \mathbf{D}^n$ is a smooth map with no fixed points. Define a smooth map $g\colon \mathbf{D}^n \to \partial\mathbf{D}^n$ as follows: for $x \in \mathbf{D}^n$, $f(x) \neq x$, so we define $g(x)$ to be the intersection point of the ray from $f(x)$ to $x$ with $\partial\mathbf{D}^n$. For $x \in \partial\mathbf{D}^n$, $g(x) = x$. This contradicts Lemma 4.

For the general case, we need the following lemma:

**Lemma 5.** *Let $f\colon \mathbf{D}^n \to \mathbf{R}^m$ be a continuous map. Then for all $\epsilon > 0$, there exists a smooth map $f_\epsilon\colon \mathbf{D}^n \to \mathbf{R}^m$ such that $||f_\epsilon(x) - f(x)|| < \epsilon$ for all $x \in \mathbf{D}^n$.*

*Proof.* If we can show it in each coordinate, we are done. So suffices to prove for $m = 1$. The Weierstraß approximation theorem says that any continuous function on an open set in $\mathbf{R}^n$ can be approximated by polynomials. □

So now assume $f\colon \mathbf{D}^n \to \mathbf{D}^n$ is continuous and, assume $f$ has no fixed points. Set

$$\delta = \inf\{||f(x) - x|| \mid x \in \mathbf{D}^n\} > 0.$$

Choose $\epsilon > 0$ much smaller than $\delta$, small enough to make the following work. Approximate $f$ by a smooth function $g\colon \mathbf{D}^n \to \mathbf{R}^n$ satisfying

$$||g(x) - f(x)|| < \epsilon$$

for all $x \in \mathbf{D}^n$. Now $g(x) \in \overline{B(0, 1+\epsilon)}$, where $B(0, 1+\epsilon)$ is the open ball centered at the origin of radius $1+\epsilon$. Define $h\colon \mathbf{D}^n \to \mathbf{D}^n$ via the formula

$$h(x) = \frac{g(x)}{1+\epsilon} \in \mathbf{D}^n.$$

Then

$$\begin{aligned} ||h(x) - x|| &\geq ||f(x) - x|| - ||\frac{g(x)}{1+\epsilon} - g(x)|| - ||g(x) - f(x)|| \\ &\geq \delta - \epsilon' - \epsilon \end{aligned}$$

where $\epsilon'$ is the maximum distance from $p$ to $\frac{p}{1+\epsilon}$. Choosing $\epsilon$ small enough, this will be positive for all $x \in \mathbf{D}^n$, which is a contradiction. □

# 6 9/10/2015: Paritions of Unity, Tubular Neighborhoods and Homotopies

We used partitions of unity when we showed you can embed smooth manifolds in some Euclidean space. Let's make this notion more precise.

**Definition 14.** Let $M^n$ be a smooth compact manifold, and let $\{U_i\}_{i=1}^k$ be a finite open cover [5]. A *smooth partition of unity* subordinate to $\{U_i\}$ is a collection of smooth functions $\{f_i\colon M^n \to \mathbf{R}\}_{i=1}^k$ such that $f_i(x) \geq 0, \mathrm{Supp}(f) \subseteq U_i, and \sum_i f_i = 1$.

**Theorem 12.** *Given any open cover $\{U_i\}_{i=1}^k$ of a compact manifold $M^n$, there exists a smooth partition of unity subordinate to $\{U_i\}_{i=1}^k$.*

This theorem is proved using the following lemma:

**Lemma 6.** *Let $M^n$ is a smooth manifold, $p \in M^n$, $U \subseteq M^n$, $U \subseteq M^n$ be an open neighborhood. Then there exists $f\colon M^n \to \mathbf{R}$ such that $f(x) \geq 0, \mathrm{Supp} f(x) \subseteq U, f|_{V_p} = 1$, where $V_p$ is a neighborhood of $p$.*

*Proof.* In real analysis, one constructs bump functions on $\mathbf{R}^n$. Just import these to a chart around $p$ contained in $U$. □

[5] in fact, one only requires that the cover be locally finite, but since we are mostly dealing with compact manifolds in this course, this definition should suffice

*Proof.* (**Theorem 12**) For $p \in M^n$, pick $i_p$ such that $p\imath U_{i_p}$. Using Lemma 6, one can find smooth functions $f_p \colon M^n \to \mathbf{R}$ such that $f_p(x) \geq 0, \mathrm{Supp}(f_p) \subseteq U_{i_p} f_p|_{V_p} = 1$ for some $V_p \subseteq U_{i_p}$. Since $M^n$ is compat, we can find $p_1, \ldots, p_\ell \in M^n$ such that $\{V_{p_j}\}_{j=1}^{\ell}$ covers $M^n$. Define

$$f_i \colon M^n \to \mathbf{R}$$

via

$$f_i = \frac{\sum_{i_{p_j}=i} f_{p_j}}{\sum_{j=1}^{\ell} f_{p_j}}.$$

It is clear that $f_i(x) \geq 0$ and $\mathrm{Supp}(f_i) \subseteq U_i$. Now we check

$$\sum_{i=1}^{k} f_i = \frac{\sum_{i=1}^{k} \sum_{i_{p_j}=i} f_{p_j}}{\sum_{j=1}^{\ell} f_{p_j}} = \frac{\sum_{j=1}^{\ell} f_{p_j}}{\sum_{j=1}^{\ell} f_{p_j}} = 1.$$

□

There are many useful corollaries. For example, we used the following when we embedded manifolds into $\mathbf{R}^n$.

**Corollary 1.** *Let $M^n$ be a smooth compact manifold, $C \subseteq M^n$ closed, $C \subseteq U$ where $U$ is open. Then there exists a smooth function $f \colon M^n \to \mathbf{R}$ such that $f(x) \geq 0$ and $\mathrm{Supp}(f) \subseteq U$ and $f|_C = 1$.*

*Proof.* Let $U' = M^n \backslash C$. $\{U, U'\}$ is an open cover, so we can find a partition of unity subordinate to this cover. □

Here's another application, generalizing a technique we used to prove the Brouwer Fixed Point theorem.

**Theorem 13.** *Let $M^n$ be a smooth compact manifold, $f \colon M^n \to \mathbf{R}^m$ continuous. Then for any $\epsilon > 0$, there exists a smooth $g \colon M^n \to \mathbf{R}^m$ such that*

$$||f(x) - g(x)|| < \epsilon$$

*for all $x \in M^n$.*

*Proof.* Choose a finite smooth atlas for $M^n$, $\{\varphi_i \colon U_i \to V_i\}_{i=1}^{k}$. Let $\{f_i \colon M^n \to \mathbf{R}\}$ be a smooth partition of unity subordinate to covering by the charts of the atlas. Define

$$h_i = f_i \cdot f.$$

Then $\text{Supp}(g_i) \subseteq U_i$. Using Stone-Weierstrass, one can find a smooth function

$$\psi_i \colon V_i \to \mathbf{R}$$

such that $\text{Supp}(\psi_i)$ is compact (this is a small extension to the regular SW theorem) and $||\psi_i(x) - h_i \circ \varphi_i^{-1}(x)|| < \epsilon/k$ (for all $x \in V_i$).

Define

$$\Lambda_i \colon M^n \to \mathbf{R}^m$$

by

$$\Lambda_i(x) = \begin{cases} \psi_i(\varphi_i(x)) & x \in U_i \\ 0 & \text{otherwise} \end{cases}$$

which is smooth. Finally, defining

$$g = \Lambda_1 + \ldots + \Lambda_k$$

we have, for $x \in M^n$

$$\begin{aligned} ||f(x) - g(x)|| &= ||(h_1(x) + \ldots + h_k(x)) - (\Lambda_1(x) + \ldots + \Lambda_k(x))|| \\ &\leq \sum_{i=1}^{k} ||h_i(x) - \Lambda_i(x)|| \leq k(\epsilon/k) = \epsilon \end{aligned}$$

□

So continuous functions from manifolds to continuous ones are nearly smooth. We want to extend this to a statement about maps from manifolds to manifolds. For this,we need the tubular neighborhood theorem

**Theorem 14. (Tubular Neighborhood Theorem)** *Let $M^n$ be a compact smooth manifold in $\mathbf{F}^m$, so $M^n \subseteq \mathbf{R}^m$. For $\epsilon > 0$ sufficiently small, one can find a small open set $U_\epsilon \subseteq \mathbf{R}^m$ containing $M^n$ and a smooth function $\pi \colon U_\epsilon \to M^n$ with the following properties*

- *$\pi(x) = x$ for all $x \in M^n$.*
- *$||\pi(x) - x|| < \epsilon$ for $x \in U_\epsilon$*

Before we prove the theorem, lets prove a corollary.

**Corollary 2.** *Let $M_1$ and $M_2$ be smooth compact manifolds. Fix a metric space structure on $M_2$ with distance $d_{M_2}$. For any continuous function $f \colon M_1 \to M_2$ and any $\epsilon > 0$, there exists a smooth function $g \colon M_1 \to M_2$ such that $d_{M_2}(f(x), g(x)) < \epsilon$ for all $x \in M_1$.*

*Proof.* Embed $M_2$ into $\mathbf{R}^m$. We can find $\epsilon' > 0$ such that, for all $x, y \in M^n$,

$$||x - y||_{\mathbf{R}^n} < \epsilon' \implies d_{M_2}(x, y) < \epsilon$$

because the metric from $\mathbf{R}^n$ restricted to $M_2^n$ and the metric $d_{M_2}$ induce the same topology. Let $\pi\colon U_{\epsilon'} \to m_2$ be an $\epsilon'$-tubular neighborhood, which exists by the theorem. Then we can find a smooth $h\colon M \to \mathbf{R}^m$ such that

$$||f(x) - h(x)||_{\mathbf{R}^m} < \frac{\epsilon'}{2}.$$

Hence, $\mathrm{Im}(h) \subseteq U_{\epsilon'}$, so we can define $g = \pi \circ h$. For $x \in M_1$, we have

$$||f(x) - g(x)||_{\mathbf{R}^m} \leq ||f(x) - h(x)||_{\mathbf{R}^m} + ||h(x) - g(x)||_{\mathbf{R}^m} \leq \frac{\epsilon'}{2} + \frac{\epsilon'}{2} = \epsilon'.$$

Thus, $d_{M_2}(f(x), g(x)) < \epsilon$. □

One more corollary, and we'll prove tubular neighborhood next time. This is a strong statement about homotopies of manifolds. First, we need a definition.

**Definition 15.** Two continuous functions $f_0, f_1\colon M_1 \to M_2$ are **homotopic** if there exists a continuous function

$$F\colon M_1 \times I \to M_2$$

such that

$$F(x, 0) = f_0(x)$$

$$F(x, 1) = f_1(x).$$

For example, any arbitrary $f_1, f_2\colon M \to \mathbf{R}^n$, we have

$$F(x, t) = (1 - t)f_1(x) + tf_2(x).$$

**Theorem 15.** *Given $M_1, M_2$ smooth compact manifolds, there exists $\epsilon > 0$ (depending on $M_2$) such that if $f_0, f_1\colon M_1 \to M_2$ are such that (for some metric $d_{M_2}$)*

$$d_{M_2}(f_0(x), f_1(x)) < \epsilon$$

*for all $x \in M_1$, then $f_0$ and $f_1$ are homotopic.*

**Corollary 3.** *Given $M_1, M_2$ compact smooth manifolds, every continuous function $f\colon M_1 \to M_2$ can be homotoped to a smooth function.*

*Proof.* Embed $M_2$ in $\mathbf{R}^m$. To simplify things, we can assume $d_{M_2}$ is induced by $||\cdot||_{\mathbf{R}^m}$. Pick $\epsilon_1 > 0$ small enough such that the tubular neighborhood $U_{\epsilon_1}$ of $M_2$ exists with projection $\pi\colon U_{\epsilon_1} \to M_2$. Next, pick $\epsilon > 0$ small enough such that for $q, p \in M_2$ with $||p-q|| < \epsilon$, the line segment $(1-t)p+tq$ in $\mathbf{R}^m$ lies in $U_{\epsilon_1}$. Now, given $f_0, f_1\colon M_1 \to M_2$ such that $||f_0(x) - f_1(x)|| < \epsilon$ for all $x \in M_1$. Define $F\colon M_1 \times I \to M_2$ by $F(x,t) = \pi((1-t)f_0(x) + tf_1(x))$.

□

A useful variant is the following theorem, proved by the same method.

**Theorem 16.** *If $f_0, f_1\colon M_1 \to M_2$ are smooth, homotopic maps between smooth manifolds, then there exists a smooth homotopy: a smooth function $F\colon M_1 \times I \to M_2$ such that $F(x,0) = f_0(x), F(x,1) = f_1(x)$ for $x \in M_1$.*

# 7 9/15/2015: More tubular neighborhoods; Degree of smooth maps

*"You don't need to respect me. In fact, I demand that you don't."*

We still need to prove the tubular neighborhood theorem. Let us recall the statement of the theorem, and perhaps restate it a little differently. We first need a definition.

**Definition 16.** Consider a compact submanifold $M^n$ of $\mathbf{R}^m$. For $p \in M^n$, we have $T_pM^n \subseteq T_p\mathbf{R}^m = \mathbf{R}^m$. The **normal bundle** of $M^n$ in $\mathbf{R}^m$, denoted $N_{\mathbf{R}^m/M^n}$, is the set

$$\{(p,n) \in T\mathbf{R}^m \mid p \in M^n \text{ and} n \text{ orthogonal to } T_pM^n \subseteq T_p\mathbf{R}^m\}.$$

In the homework, we showed that $TM^n$ is a $2n$-dimensional manifold with projection $TM^n \to M^n$, a submersion. A similar argument shows that $N_{\mathbf{R}^m/M^n}$ is a $(n + (m-n)) = m$-dimensional submanifold and the projection $N_{\mathbf{R}^m/M^n} \to M^n$ taking $(p,n) \mapsto p$ is a submersion.

For example, consider $S^n \subseteq \mathbf{R}^{n+1}$. Then $TS^n = \{(p,v) \in T\mathbf{R}^{n+1} \mid v \perp p\}$. The normal bundle is $\{(p,n) \in T\mathbf{R}^{n+1} \mid v = \lambda p\}$

Let us introduce some notation. Fix a submanifold $M^n \subseteq \mathbf{R}^m$. Define

$$\psi\colon N_{\mathbf{R}^m/M^n} \to \mathbf{R}^m$$

via

$$\psi(p,n) = p + n.$$

For $\epsilon > 0$, define

$$N^\epsilon = \{(p,n) \in N_{\mathbf{R}^m/M^n} \mid ||n|| < \epsilon\}.$$

Then we have a submersion $\pi_\epsilon \colon N^\epsilon \to M^n$.

**Theorem 17. (Tubular neighborhood, revisited)** *For $\epsilon > 0$ small enough, $\psi|_{N^\epsilon}$ is an embedding.*

*Proof.* **Step 1:** We show the following: there exists an open cover $\{U_i\}_{i=1}^k$ of $M^n$ such that, defining $N^\epsilon(U_i) = \pi_\epsilon^{-1}(U_i)$, the map $\psi|_{N^\epsilon(U_i)}$ is an embedding for all $i$ for all $\epsilon > 0$ sufficiently small.

Note that it is enough to show that for $p \in M^n$, the map $\psi$ is a local diffeomorphism at $(p, 0)$. Then we use compactness to find a finite subcover and small enough $\epsilon$ that works for everything. Note that

$$D_{(p,0)}\psi \colon T_{(p,0)}N_{\mathbf{R}^m/M^n} \to T_p\mathbf{R}^m = \mathbf{R}^m$$

is simply the identity map.

**Step 2:** For $\epsilon > 0$ even smaller, we have

$$\psi(N^\epsilon(U_i)) \cap \psi(N^\epsilon(U_j)) = \psi(N^\epsilon(U_i \cap U_j)).$$

Indeed, for any two $i, j$, we can clearly choose $\epsilon$ small enough so that this works. Just take the minimum $\epsilon$ over all $i, j$.

**Step 3:** $\psi|_{N^\epsilon}$ is a local diffeomorphism by Step 1, which is injective (by Step 2). Hence, $\psi$ is an embedding. □

**Theorem 18.** *$M^n$ is a connected, smooth manifold, $p, q \in M^n$. Then there exists a diffeomorphism $f \colon M^n \to M^n$ such that $f(p) = q$.*

*Proof.* Perhaps a more elegant proof of this fact is given by looking at flows on $M^n$. We have yet to discuss vector fields, so we give an alternate proof, using flows on the disc. Let

$$\Lambda = \{x \in M^n \mid \text{there exists a diffeomorphism } f \colon M^n \to M^n, f(p) = x\}.$$

Since $M^n$ is connected, it is enough to show $\Lambda$ is open and closed.

The key to the proof is the following claim:

*Given $x, y \in \mathbf{D}^n$, $x, y \notin \partial\mathbf{D}^n$, there exists a diffeomorphism $g \colon \mathbf{D}^n \to \mathbf{D}^n$ such that $g(x) = y$ and $g$ restricts to the identity in a neighborhood of $\partial D^n$.*

To prove this claim, one can find an embedding $\gamma\colon [0,1] \to \mathrm{Int}(\mathbf{D}^n)$ whose image is a straight line connecting $x$ and $y$. We get a vector field $\mathrm{Im}(\gamma) \to \mathbf{R}^n$ from the ordinary derivative of $\gamma$. One can extend to a smooth vector field $\eta$ on $\mathbf{D}^n$ such that $\eta$ is 0 on neighborhood of $\partial\mathbf{D}^n$. Then flow in the direction of $\gamma$. This gives the $g$ we want.

Now we prove that $\Lambda$ is open. Consider $x \in \Lambda$. One can find a closed $C \subseteq M^n$ diffeomorphic to $\mathbf{D}^n$ containing $x$ in its interior. Consider $y \in \mathrm{Im}(\gamma) \subseteq \mathrm{Int}(C)$. Using the claim proved above, we can find a diffeomorphism $g\colon C \to C$ with $g(x) = y$ and $g|_{\text{nbdh of } \partial C} = \mathrm{id}$. Extend $g$ by id to $\hat{g}\colon M^n \to M^n$, then $\hat{g}(x) = y$, so $\mathrm{Int}(C) \subseteq \Lambda$.

Now we show $\Lambda$ is closed. Consider $y \in \overline{\Lambda}$. We can find a closed disc $C \subseteq M^n$ diffeomorphic to $\mathbf{D}^n$ such that $y \in \mathrm{Int}(C)$. Pick $x \in \Lambda \cap \mathrm{Int}(C)$. An argument like the previous claim produces $g\colon M^n \to M^n$ such that $\hat{g} = y$, and hence $y \in \Lambda$.

□

**Corollary 4.** *Given a continuous map $f\colon M_1 \to M_2$, and $p \in M_2$, there is a smooth map $g\colon M_1 \to M_2$ homotopic to $f$ such that $p$ is a regular value of $g$.*

*Proof.* From before, we can homotope $f$ to a smooth map $g_1\colon M_1 \to M_2$. Sard says we can find a regular value $q$ of $g_1$. The theorem we just proved says that we can find a family $h_t\colon M_2 \to M_2$ such that $h_0 = \mathrm{id}$ and $h_1$ is a diffeomorphism with $h_1(p) = q$. Then $\varphi_t = h_t \circ g_1$ is a family of smooth maps $M_1 \to M_2$ with $\varphi_0 = g_1$ and $\varphi_1 = h_1 \circ g_1$. Since $h_1(p) = q$, and $q$ is a regular value of $g$, $p$ is a regular value of $\varphi$. □

**Definition 17.** Let $M_1, M_2$ be smooth compact manifolds of the same dimension and let $f\colon M_1 \to M_2$ be a continuous map. The **mod-2 degree** of $f$ is

- Pick a smooth map $g\colon M_1 \to M_2$ homotopic to $f$.
- Pick regular value $p \in M_2$
- Then $\deg f = |g^{-1}(p)| \mod 2$.

**Theorem 19.** *This is well-defined.*

**Corollary 5.** *Let $M$ be a smooth compact manifold. Then $\mathrm{id}\colon M \to M$ is not homotopic to a constant map.*

*Proof.*

$$\deg(\mathrm{id}) = 1 \mod 2$$

$$\deg(\text{constant map}) = 0 \mod 2$$

□

# 8 9/17/2015

Let us give a result which follows from the lemmas of last time.

**Lemma 7.** *Let $f_0, f_1$ be homotopic smooth maps, $p \in M_2$, $p$ a regular value of $f_0$ and $f_1$. Then there exists a smooth $F\colon M_1 \times I \to M_2$ such that $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$, and $p$ is a regular value of $F$.*

Let us now state and prove our big theorem about degree mod 2.

**Theorem 20.** *Let $f_1\colon M_1^n \to M_2^n$ be a continuous function between compact, connected manifolds of the same dimension. Pick $g\colon M_1^{n_1} \to M_2^{n_2}$ smooth, homotopic to $f$, and $p \in M_2^n$ a regular value of $g$. Define $\deg_2(f) = |g^{-1}(p)| \mod 2$. This is well defined (independent of $g$ and $p$).*

*Proof.* First, we how that $h\colon M_1^n \to M_2^n$ is smoothly homotopic to $g$, and has regular value $p$. Then we wish to show that $|g^{-1}(p)| = |h^{-1}(p)| \mod 2$. Using the lemma above, find smooth $G\colon M_1^n \times I \to M_2^n$ with $G(x,0) = g(x), G(x,1) = h(x)$, and $p$ is a regular value of $G$. Then $G^{-1}(p)$ is a compact 1-manifold with boundary in $M_1^n \times I$ such that $\partial(G^{-1}(p)) = G^{-1}(p) \cap \partial(M_1^n \times I)$. So these are either circles, an interval connecting two points of either $g^{-1}(p) \times \{0\}$ or $h^{-1}(p) \times \{1\}$, or an interval connecting a point of $g^{-1}(p) \times \{0\}$ to a point of $h^{-1}(p) \times \{1\}$. Every point of $g^{-1}(p) \times \{0\}$ and $h^{-1}(p) \times \{1\}$ is the endpoint of some component of $G^{-1}(p)$. An even number of points of $g^{-1}(p) \times \{0\}$ are endpoints of the intervals connecting two points on one boundary piece of $M_1^n \times I$. The same number of points for every 1-manifold of the third type contribute the same number to the count for $|g^{-1}(p)|$ and $|h^{-1}(p)|$.

Next, we need to show that the degree is independent of $p$. Use the fact that given any $p$ and $q$, we can find a family $\eta_t\colon M_2 \to M_2$ of a diffeomorphism with $\eta_0\colon \mathrm{id}$ and $\eta_1(p) = q$. Thus, if $p$ is a regular value of $g$, then $q$ is a regular value of $g \circ \eta_1$, and $g \circ \eta_1$ is homotopic to $g$ (and to $f$) and $g^{-1}(p) = (g \circ \eta_1)^{-1}(q)$.

□

To refine this notion of degree to get a number in $\mathbf{Z}$, we need to chosse a sign $\epsilon_x$ for each $x \in g^{-1}(p)$ such that a point belonging to an interval starting and ending from the same side contributes $-1$, and a point belonging to an interval connecting points of each boundary component contributes 1. If we could do this, we would define

$$\deg(f) = \sum_{x \in g^{-1}(p)} \epsilon_x.$$

In order to do this, we need the notion of orientation.

**Definition 18.** Let $\mathcal{B}_n = \{$ordered bases $(b_1, \ldots, b_n)$ for $\mathbf{R}^n\}$. We say that $(b_1, \ldots, b_n) \sim (c_1, \ldots, c_n)$ if the matrix $M$ with $M(b_i) = c_i$ has $\det M > 0$. An **orientation** of $\mathbf{R}^n$ is an element of $\mathcal{B}_n / \sim$. Note that there are two orientations.

Given $\sigma \in S_n$, we have $(b_{\sigma(1)}, \ldots, b_{\sigma(n)}) = (b_1, \ldots, b_n)$ if $\operatorname{sign}(\sigma) = 1$.

Another point of view is through the identification $\Lambda^n \mathbf{R}^n \simeq \mathbf{R}^1$. Given a basis $(b_1, \ldots, b_n)$ for $\mathbf{R}^n$, we have $b_1 \wedge \cdots \wedge b_n \in \Lambda^n \backslash \{0\}$. This has two connected components. The component it lands in is the orientation.

The informal definition for **orientation of a manifold** is a consistent choice of orientation for each $T_p M^n$ which "varies continuously". The formal definition is given below.

**Definition 19.** An **oriented smooth manifold** is a smooth $M^n$ equipped with a smooth atlas $\{\varphi_i \colon U_i \to V_i\}_{i \in I}$ such that the determinants of derivatives of transition functions are positive, i.e. for all $i, j \in I$ and $p \in U_i \cap U_j$, we have

$$\det(D_{\varphi(p)} \tau_{ji} \colon T_{\varphi_i(p)} V_i \to T_{\varphi_j(p)} V_j) > 0$$

$\mathbf{R}^n$ has the "standard orientation" corresponding to the standard basis. One can assign this to each tangent space of $V_i$ since $T_q V_i = \mathbf{R}^n$. If $M^n$ is an oriented manifold, the restriction of the derivative of transition functions implies that the above gives a consistent choice of orientation on each $T_p M^n$.

**Definition 20.** Let $f \colon M_1^n \to M_2^n$. be a smooth map between oriented manifolds of the same dimension, $p \imath M_2^n$ a regular value. For $x \in f^{-1}(p)$, we have

$$D_x f \colon T_x M^n \to T_p M_2^n.$$

We say that $\epsilon_x = 1$ if $D_x f$ takes an orientation of $T_x M_1^n$ to an orientation of $D_p M_2^n$, $\epsilon_x = -1$ if it does not.

The Mobius band is an example of a nonorientable manifold.

**Lemma 8.** *Given a smooth function $f\colon \mathbf{R}^{n+1} \to \mathbf{R}$ and a regular value $p \in \mathbf{R}$. Then $f^{-1}(p)$ is orientable.*

*Proof.* Consider $x \in f^{-1}(p)$. We need to choose an orientation on $T_x f^{-1}(p) = \ker(D_x f\colon T_x\mathbf{R}^{n+1} \to T_p\mathbf{R})$.

We can choose a basis $\{b_1, \dots, b_{n+1}\}$ for $\mathbf{R}^{n+1}$ such that the following holds:

- $(b_1, \dots, b_{n+1})$ gives the standard orientation on $\mathbf{R}^{n+1}$.
- $(b_1, \dots, b_n)$ is a basis for $T_x f^{-1}(p)$.
- $(D_x f)(b_{n+1}) > 0$.

An easy exercise is to show that $(b_1, \dots, b_n)$ gives a well-defined orientation to $T_x f^{-1}(p)$ that varies continuously. □

**Lemma 9.** *Let $M$ be an oriented manifold with boundary. Then one can orient $\partial M^n$.*

*Proof.* For $p \in \partial M$, pick a basis $(b_1, \dots, b_n)$ for $T_pM$ such that

- $(b_1, \dots, b_{n-1})$ is a basis for $T_p\partial M^n$
- $(b_1, \dots, b_n)$ gives an orientation on $M^n$
- $b_n$ faces into $M^n$.

□

## 9 9/22/2015: Orientation and Degree

Recall that an orientation for a finite dimensional $\mathbf{R}$-vector space is a choice of basis $(v_1, \dots, v_n)$, modulo the following equivalence:

$$(v_1, \dots, v_n) \sim (w_1, \dots, w_n) \iff \det A > 0, A := (a_ij), w_i = \sum_i a_{ij} v_j.$$

An orientation on a smooth manifold $M^n$ is a smoothly varying choice of orientation on each $T_pM^n$, where smoothly varying means that if $\varphi\colon U \to V$ is a chart, then for every $p \in U$, the induced orientation on each $T_{\varphi(p)}V = \mathbf{R}^n$ is the same, i.e. the identification $D_p\varphi\colon T_pM^n \cong T_{\varphi(p)}V = .\mathbf{R}^n$ induces the same orientation on $\mathbf{R}^n$ for every $p \in U$.

For example, $S^n (\subseteq \mathbf{R}^{n+1})$ is orientable. We have $T_pS^n = \{v \in T_p\mathbf{R}^{n+1} \mid v \text{orthogonal to line from 0 to } p\}$. The orientation on $T_pS^n$ corresponding to basis $(v_1, \ldots, v_n)$ for $T_pS^n$ is such that the basis $(v_1, \ldots, v_n, p)$ is the standard orientation on $\mathbf{R}^{n+1}$. This works by the last homework problem this week; namely, if $x$ is a vector space and $X = Y \oplus Z$, then, given an orientation on two of $X, Y$, and $Z$, there is a unique orientation on the third such that if $(y_1, \ldots, y_k)$ is an oriented basis of $Y$, and $(z_1, \ldots, z_\ell)$ is an oriented basis for $Z$, then $(y_1, \ldots, y_k, z_1, \ldots, z_\ell)$ is an oriented basis for $X$ (just linear algebra). We call this the "two out of three" argument.

More generally, we have the following lemma.

**Lemma 10.** *Let $f\colon M_1 \to M_2$ be a smooth map of smooth manifolds, $p \in M_2$, $p$ a regular value, $M_1$ oriented. Then $f^{-1}(p) = A$ is oriented.*

*Proof.* For $q \in A$, we have

$$T_qA = \ker(T_qM_1 \to T_pM_2).$$

Fix some orientation on $T_pM_2$. Then there is a short exact sequence

$$0 \to T_qA \to T_qM_1 \to T_pM_2 \to 0$$

because $p$ is regular, so we get an induced orientation on $T_qA$

□

Recall the non-example of the Mobius band. Another non example is $\mathbf{R}P^2 = S^2/\sim$, where $\sim$ identifies antipodal points. This is because $\mathbf{R}P^2$ contains a Mobius band, so an orientation on $\mathbf{R}P^2$ would induce an orientation on the Mobius band, which is impossible.

Let us now recall the construction of the induced orientation on the boundary. Let $M^n$ be a smooth orientated manifold with boundary. We want to construct an orientation $\partial M^n$. For each $p \in \partial M^n$, we can write

$$T_pM^n = T_p(\partial M^n) \oplus \langle v \rangle$$

where $v$ points inward. The "two out of three" argument says that the given orientation on $T_pM^n$ gives a unique orientation of $T_p(\partial M^n)$ such that if $(b_1, \ldots, b_{n-1})$ is an oriented basis for $T_p(\partial M^n)$ and $v \in T_pM^n$ points inward, then $(b_1, \ldots, b_{n-1}, v)$ is an oriented basis. Call this the **inward facing orientation** on $\partial M^n$. One could have also constructed the **outward facing orientation** on $\partial M^n$.

**Definition 21.** Let $M_1^n$ and $M_2^n$ be oriented, compact, connected, smooth $n$-manifolds, $f\colon M_1^n \to M_2^n$ be continuous. The **degree** of $f$, denoted $\deg(f) \in \mathbf{Z}$, is

- Pick a smooth map $g\colon M_1^n \to M_2^n$, homotopic to $f$.
- Pick a regular value $p \in M_2^n$ of $g$.
- Define $\deg(f) = \sum_{q\in f^{-1}(p)} \epsilon_q$

where $\epsilon_q = 1$ if the isomorphism $D_q g\colon T_q M_1^n \to T_p M_2^n$ preserves orientation, and $\epsilon_q = -1$ if it does not.

**Theorem 21.** *The above definition does not depend on choice of $g$ or $p$.*

*Proof.* Following the proof for mod 2 degree, it is enough to prove the following:

*If $g_0, g_1\colon M_1 \to M_2$ are smooth with $p \in M_2$ a regular value and $F\colon M_1 \times I \to M_2$ is a smooth homotopy from $g_0 \to g_1$ with $p$ a regular value of $F$, then*

$$\sum_{q\in g_0^{-1}(p)} \epsilon_q = \sum_{q\in g_1^{-1}(p)} \epsilon_q$$

Like our proof for mod 2 degree, $F^{-1}(p)$ is a compact 1-manifold in $M_1 \times I$ such that $\partial F^{-1}(p) = g_0^{-1}(p) \times \{0\} \cup g_1^{-1}(p) \times \{1\}$. Once again, we have three kinds of components of $F^{-1]}(p)$:

- circles in interior
- arcs connecting points on the same side (arc of the second kind)
- arcs connecting point of one side to point of the other side (arc of the third kind)

If an arc of the second kind connects $q_1, q_2 \in g_0^{-1}(p)$, then we claim that $\epsilon_{q_1} = -\epsilon_{q_2}$. Also, we claim that if an arc of the third kind connects $q_1 \in g_0^{-1}(p)$ and $q_2 \in g_1^{-1}(p)$, then $\epsilon_{q_1} = \epsilon_{q_2}$. Proving these two claims will complete the proof of the theorem.

We can choose an orientation on $M_1 \times [0,1]$ such that the inward orientation on $M_1 \times \{0\}$ is the chosen orientation on $M_1$. Note that the inward orientation on $M_1 \times \{1\}$ is opposite the orientation on $M_1$. Our orientation on $M_1 \times [0,1]$ induces an orientation on $F^{-1}(p)$: for $r \in F^{-1}(p)$, since $p$ is a regular value, we have a short exact sequence

$$(*)\quad 0 \to T_r F^{-1}(p) \to T_r(M_1 \times [0,1]) \to T_p M_2 \to 0$$

so we get an orientation on $T_r F^{-1}(p)$. On an arc of the second kind $q_1 \to q_2$, the orientation on the arc faces inwards at one point, outwards at the other. Say it faces inwards at $q_1$, outwards at $q_2$. Then to see that $\epsilon_{q_1} = -\epsilon_{q_2}$, observe that by changing the orientation on $M_2$, we can assume that $\epsilon_{q_1} = 1$. But then $M_1 \times [0,1]$ is oriented such that the induced orientation on each point of the arc is from (*). So we see that on the other endpoint $q_2$ of the arc, the map $g_0$ preserves orientation, but with outward orientation. But from the definition of $\epsilon$, $\epsilon_{q_1} = -\epsilon_{q_2}$.

The same argument yields $\epsilon_{q_1} = \epsilon_{q_2}$ for arcs of the second kind. □

## 10 9/24/2015: Applications of degree

In general, it is hard to determine what possible degrees occur among continuous maps $f\colon M_1^n \to M_2^n$. However, this is completely understood if $M_2^n = S^n$. For example, we can build a degree 1 map $f\colon M_1^n \to \mathbf{R}^n \cup \{\infty\}$. Fix a small disc $D$ in $M_1^n$. $f$ takes the interior of $D$ onto $\mathbf{R}. = S^n \backslash \{\infty\}$, preversing orientation, and take $M_n \backslash \mathrm{Int}(D)$ to $\infty$. This gives a continuous map $f\colon M_1^n \to S^n$, and, if we are careful, a smooth map. All points but $\infty$ are regular values with one preimage $\deg(f) = 1$.

We can also use this kind of construction to get a degree $k\mathrm{Im}\mathbf{Z}$ map $f\colon S^n \to S^n$. Choose $|k|$ disjoint discs $D_1 \ldots, D_k$ and do the same thing. $f$ takes each $\mathrm{Int}(D_i)$ diffeomorphically onto $S^n \backslash \{\infty\}$, preserving or reversing orientation depending on the sign of $k$. $f$ take $\bigcup_{i=1}^{|} k| \backslash \mathrm{Int}(D_i)$ to $\infty$.

We now set out to prove the following remarkable theorem.

**Theorem 22. (Hopf Degree Theorem):** *Given any compact oriented $n$-manfiold and $f, g\colon M^n \to S^n$, then $f$ is homotopic to $g$ if and only if* $\deg(f) = \deg(g)$.

**Remark 2.** If $m < n$, all $f\colon M^n \to S^n$ are homotopic to constant maps; homotope to a smooth map, let $p \in S^n$ be a regular value, then $f^{-1}(p) = \emptyset$. So, $\mathrm{Im}(f) \subset S^n \backslash \{p\} \simeq \mathbf{R}^n$. Use the straight line homotopy to homotope $f$ to a constant map.

**Remark 3.** If $m > n$, it is **much** harder to understand homotopy classes of maps $M^n \to S^n$, and in most cases there are little known. For example, for $k \geq 200$, it is not known how many homotopy classes of maps of $S^{n+k} \to S^n$, but Serre proved in his thesis that there are finitely many unless $n + k = 2n - 1$.

*Proof.* **(sketch):** Given $f, g\colon M^n \to S^n$ smooth, and $\deg(f) = \deg(g)$, we can assume that $p = (0, 0, \ldots, -1) \in S^n$ is a regular value of $f, g$. We want to show that we can homotope $f$ such that it looks like our description of a degree $k$ map like we constructed above. Write $f^{-1}(q) = q_1, \dots, q_\ell$. We know that $f$ is a local diffeomorphism at each $q_i$ (since $p$ is a regular value). Thus, we can find small discs $D_1, \dots, D_\ell$ in $M_1$ and a small disc $E \in S^n$ such that

- $q_i$ is the center of $D_i$
- $p_i$ is the center of $E$.
- $f|_{D_i}$ is a diffeomorphism onto $E$, preserving/reversing orientation depending on $\epsilon_{q_i}$.

Our first claim is that we can homotope $f$ such that $E = S^n \backslash \{\infty\}$ and $f(x) = \infty$ for $x \in M^n \backslash (\bigcup_{i=1}^{\ell} D_i)$.

To prove this claim, we remark that we can find a family of smooth maps $\varphi_i \colon S^n \to S^n$ such that $\varphi_0 = \mathrm{id}$ and $\varphi_1$ takes $E$ diffeomorphicall onto $S^n \backslash \{\infty\}$ and $S^n \backslash \mathrm{Int}(E)$ to $\infty$. Then $\varphi_1 \circ f$ is homotopic to $f$ and has the desired properties.

Now we claim that is $\epsilon_{q_i} = -\epsilon_{q_j}$, then we can homotope $f$ so as to move $D_i$ close to $D_j$, make the collide, and cancel.

The cancellation looks like the following: Say

$$\psi_1 \colon [0,1]^2 \to \mathbf{R}^2$$

$$(x, y) \mapsto (x, y)$$

$$\psi_2 \colon [1,2] \times [0,1] \to \mathbf{R}^2$$

$$(x, y) \mapsto (2 - x, y)$$

$\psi_1$ preserves orientation, and $\psi_2$ reverses orientation. We define

$$\psi \colon [0,2] \times [0,1] \to \mathbf{R}^2$$

$$\psi|_{[0,1]^2} = \psi_1, \psi|_{[1,2]\times[0,1} = \psi_2.$$

We can deform $\psi$ to the constant map, setting $\psi_t$ to be the result of "folding" $\psi_2$ over $\psi_1$. Do the same to $g$, then homotope $g$ to move its discs to discs of $f$ and make it the same on these discs. □

There is a nice property of degree. If $f \colon M_1^n \to M_2^n$ and $g \colon M_2^n \to M_3^n$, all compact, oriented smooth manifolds, then

$$\deg(g \circ f) = \deg(g) \deg(f).$$

Here's a calculation. Define for $1 \leq i \leq n+1$,

$$f_i \colon S^n \to S^n$$

$$f_i(x_1, \ldots, x_{n+1}) = (x_1, \ldots, -x_i, \ldots, x_{n+1}).$$

Then $\deg(f_i) = -1$, since $f_i$ is an orientation-reversing diffeomorphism. All points $p$ are regular values, and $f_i^{-1}(p) = \{q\}$, and $\epsilon_q = -1$.

This implies the following lemma

**Lemma 11.** *Let $g \colon S^n \to S^n$ be the antipodal map $g(x_1, \ldots, x_{n+1}) = (-x_1, \ldots, -x_{n+1})$. Then* $\deg(g) = (-1)^{n+1}$.

*Proof.* $g = f_0 \circ \cdots \circ f_{n+1}$. □

**Corollary 6.** *If $n$ is even, then the antipodal map on $S^n$ is not homotopic to the identity.*

**Definition 22.** Let $M^n$ be a smooth manifold. A **vector field** on $M^n$ is a continuous function $\tau \colon M \to TM$ such that $\tau(p) \in T_p M^n$ for all $p$.

We now state and prove the Hairy Ball Theorem.

**Theorem 23. (Hairy Ball Theorem)** *If $\tau$ is a vector field on $S^{2n}$, then $\tau$ has a zero.*

**Remark 4.** This is false for $S^{2n+1}$. Recall that $T_p S^m$ consists of vectors in $T_p \mathbf{R}^{m+1} = \mathbf{R}^{m+1}$ orthogonal to $p$. Define

$$\tau(x_1, \ldots, x_{2n+2}) = (x_2, -x_1, x_4, -x_3, \ldots, x_{2n+2}, -x_{2n+1}).$$

*Proof.* It is enough to show that if $\tau$ is a nonvanishing vector field on $S^m$, we can use $\tau$ to construct a homotopy from id to the antipodal map. Define

$$F \colon S^n \times [0,1] \to S^n$$

as follows: consider $p \in S^n$. We can find a unique great circle $\gamma_p$ through $p$ in the direction $\tau(p)$. We can parametrize $\gamma_p$ as $\gamma_p \colon [0,1] \to S^n$ such that

- $\gamma_p(0) = \gamma_p(1) = p$.
- $\gamma_p$ moves at constant speed, i.e. $||\gamma_p'(t)||$ is constant, norm is from $\mathbf{R}^{m+1}$.

We now define $F(p,t) = \gamma_p(\frac{t}{2})$. Then

$$F(p,0) = \gamma_p(0) = p$$

$$F(p,1) = \gamma_p(1/2) = \text{halfway around great circle, i.e } -p$$

and this homotopes the identity to the antipodal map, a contradiction. $\square$