MATH 566 - SPRING 2017

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These notes cover some basic topics in the theory of *derived categories of coherent sheaves*. The initial goal was to provide a minimum background to start reading directly (and without feeling too lost) Chapter 4 of Huybrechts's most awesome "Fourier-Mukai transforms in algebraic geometry". That chapter is where the material stops being just homological algebra and starts being fun.

Presently, this pdf contains a bit more than that, reflecting essentially all the material covered in class. The exposition mainly follows Kashiwara-Schapira "Sheaves on manifolds" for the first half and Huybrechts for the second half.

I've tried to keep the exposition concise and the prerequisites at a minimum. This means that some parts are incomprehensible to the novice but also super-boring for the expert (this is especially true for the treatment of sheaves).

In closing, two things: comments about the content are always very welcome, so please feel free to email me if you have any; one may safely skip anything marked "digression".

Contents

1. Introduction

The aim of this lecture is to provide some motivation for studying derived categories. It is not important to grasp all (or any) of the details: they will become clear only after digesting material which comes much later on.

Derived categories were introduced by Verdier around 1960, while he was a student of Grothendieck. The main motivation was to provide a more convenient framework to perform homological algebra. This lead to clarifying theorems which, up to that point, could only be stated in the language of spectral sequences. Moreover, Verdier's theory lead to vast generalizations of Poincaré duality in topology and Serre duality in algebraic geometry.

The axiomatics of derived categories are a special case of triangulated categories, which had been independently developed by Dold and Puppe to uncover the structure of the stable homotopy category of spectra in topology.

Nowadays, derived and triangulated categories appear also in other fields, such as algebra (Morita theory), symplectic geometry (via Fukaya categories), representation theory (D-modules, perverse sheaves) and even string theory (branes).

We give below three examples for why one might care about derived categories. The first is mirror symmetry, which is possibly the most exciting and ambitious. The second is Serre's intersection formula, which is both technical in nature (explaining why a module should be identified with its resolutions) but also conceptual, as it leads to what is now called *derived algebraic geometry*. The final reason is upgrading spectral sequences, which is a purely technical reason and close to the origins of the subject.

1.1. Mirror symmetry. — We start with a few words on what has probably been the biggest driving force in the study of derived categories of coherent sheaves: homological mirror symmetry. We won't even be close to doing justice to the subject, but will content ourselves with a general and vague picture. An excellent source of information is Nick Sheridan's website.

Mirror symmetry actually originates in physics. The starting point are so-called N=2 superconformal field theories, which still elude a mathematically rigorous definition. These theories admit what are called topological twists: an A-model and a B-model. Since both models come from the same conformal field theory, one expects outputs of the A side to be related to those from the B side.

The A-model lives in the symplectic world. For example, if $X^\vee$ is a symplectic manifold, its Gromov–Witten invariants are A-model invariants. The B-model lives in algebraic geometry (well, possibly Kähler geometry is more correct). If X is an algebraic variety, its Hodge structure or periods of differential forms are B-model invariants.

What caught the attention of mathematicians were striking computations done by Candelas-de la Ossa-Green-Parkes, who (correctly) predicted enumerative invariants of

Calabi-Yau threefolds which were previously unknown. Specifically, they predicted the number of rational curves on a degree five hypersurface in $\mathbf{P}^4$ (the quintic threefold). A mathematically rigorous (but still mirror-symmetry-inspired) argument was later given by Givental and Lian-Liu-Yau.

In 1994, Kontsevich conjectured his homological mirror symmetry, which categorifies the relation between the A and B models and aims at explaining other examples of A and B model invariants being identified under mirror symmetry. Vaguely, he predicts that for X a Calabi-Yau threefold there should exist a mirror Calabi-Yau threefold $X^\vee$ such that D(X), the bounded derived category of coherent sheaves on X, is equivalent to DFuk($X^\vee$), the derived Fukaya category of $X^\vee$. The former will be the object of study of these notes and represents the B side. The objects of D(X) are chain complexes of holomorphic vector bundles. The latter lives in the realm of symplectic geometry and represents the A side. Its objects are, essentially, Lagrangians submanifolds. This conjecture is - to say the least - remarkable, as it bridges (in a rather surprising way!) two distinct areas of math.

1.2. Tor. — Let's move on to something more concrete: Tor, the derived functor of tensor product. Consider $X = \mathbf{C}^2 = \mathbf{A}^2$ affine two-space, a parabola $P \subset X$ given by the equation $y = x^2$, a line $L_c \subset X$ given by the equation $y = c^2$. The intersection $P \cap L_c$ is zero-dimensional and consists of the points (c, c^2) and $(-c, c^2)$. We notice immediately that, if $c \neq 0$, these are two distinct points so that $|P \cap L_c| = 2$, while when $c = 0$ we only get the origin 0. This feels wrong: the number of intersection points should be unaffected by small perturbations of our initial setup. The fix is easy enough, by using a little algebra: instead of taking the set-theoretic intersection we should be taking the *scheme*-theoretic intersection.

Let's do this algebra. Set $A = \mathbf{C}[x, y]$, so $X = \operatorname{Spec} A$, $P = \operatorname{Spec} A/(y - x^2)$, $L_c = \operatorname{Spec} A/(y - c^2)$. The intersection $P \cap L_c$ is the fibre product $P \times_X L_c$, which corresponds to the tensor product $A/(y - x^2) \otimes_A A/(y - c^2)$. We have,

$$A/(y - x^2) \otimes_A A/(y - c^2) = A/(y - x^2, y - c^2) \tag{1.1}$$

$$= \mathbf{C}[x]/(c^2 - x^2) \tag{1.2}$$

$$= \mathbf{C}[x]/(c - x)(c + x) \tag{1.3}$$

when $c \neq 0$, the remainder theorem tells us that this is isomorphic to $\mathbf{C}[x]/(c - x) \times \mathbf{C}[x]/(c + x)$ and thus $P \cap L_c = \operatorname{Spec}(\mathbf{C}[x]/(c - x)) \cup \operatorname{Spec}(\mathbf{C}[x]/(c + x))$. On the other hand, if $c = 0$, we get the ring $\mathbf{C}[x]/(x^2)$. This ring, which is called the ring of *dual numbers*, is *not* the product of two rings. However, $\dim_{\mathbf{C}} \mathbf{C}[x]/(x^2) = 2$, recovering our intersection number.

Let's see a simpler example. Take $X = \mathbf{C}^2$, Y the union of $\{x - y = 0\}$ and $\{x + y = 0\}$, Z_c the line $y = c$. Again, for $c \neq 0$ we have two points, while for $c = 0$ we get a single point with multiplicity two. Let's do the algebra.

$$A/(x - y)(x + y) \otimes A/(y - c) = \mathbf{C}[x]/(x - c)(x + c) \tag{1.4}$$

which reduces to the same example as before.

Let's have a look at a variant of the example above. Consider $X = \mathbf{C}^4$, Y the union of the 2-planes $\{x - y = 0 = z - w\}$, $\{x + y = 0 = z + w\}$ and Z_c the plane $\{y = c = w\}$. The (scheme-theoretic) intersection $Y \cap Z_c$ is given by the (reduced) points (c, c, c, c) and

$(-c, c, -c, c)$. So we expect that when $c = 0$, the intersection will have multiplicity two. Let $A = \mathbf{C}[x, y, z, w]$. The algebra tells us that

$$\frac{A}{(x-y, z-w)\cdot(x+y, z+w)} \otimes A/(y, w) = \mathbf{C}[x, z]/(x, z)\cdot(x, z) \tag{1.5}$$

$$= \mathbf{C}[x, z]/(x^2, xz, z^2) \tag{1.6}$$

which has basis (as a $\mathbf{C}$-vector space) $\{1, x, z\}$ hence has dimension three! So the scheme-theoretic intersection does not contain enough information to get the correct answer. So, what's the solution?

Serre solved this problem beautifully, by giving an algebraic expression for the correct intersection multiplicity, which involves Tor modules. Nowadays, we have an even more conceptual answer. Instead of taking the scheme-theoretic intersection of Y and Z one should take the *derived* intersection. The idea is that the structure sheaf of this derived intersection is the derived tensor product $M \overset{L}{\otimes} N$, where $M = A/(x-y, z-w)\cdot(x+y, z+w)$ and $N = A/(y, w)$. This is a chain complex of A-modules, whose cohomology are the Tor-modules: $H^{-i}(M \overset{L}{\otimes} N) = \mathrm{Tor}_i^A(M, N)$. Giving this intersection a geometric meaning involves techniques which go beyond these lectures. But let's see this circle of ideas in action in this example.

Serre tells us that the correct intersection multiplicity is given by

$$\chi(M, N) = \sum_i (-1)^i \dim_{\mathbf{C}} \mathrm{Tor}_i^A(M, N). \tag{1.7}$$

The recipe to compute Tor is more or less straightforward: we need to resolve M or N by free modules and then apply the tensor product.

Let's call $M = A/(x-y, z-w)\cdot(x+y, z+w)$, $N = A/(y, w)$. The module N, called the *Koszul resolution.* Let's construct it by hand. The module N is generated by the element 1, so $A \to N$ sending 1 to 1 is surjective.

$$0 \to \langle y, w\rangle \to A \to N \to 0 \tag{1.8}$$

Its kernel is precisely the ideal $\langle y, w\rangle$. Thus we have a surjective map $A^{\oplus 2} \to \langle y, w\rangle$ sending $(1, 0)$ to y and $(0, 1)$ to w. More concisely, the vector (f, g) is sent to $yf + wg$.

$$0 \to K \to A^{\oplus 2} \to A \to N \to 0 \tag{1.9}$$

One checks that the kernel K of this map is generated by the obvious candidate: the vector $(w, -y)$. So we have a surjective map $A \to K$ taking 1 to $(w, -y)$. One checks that this map is injective, leading to our free resolution

$$0 \to A \xrightarrow{\begin{pmatrix} w \\ -y \end{pmatrix}} A^{\oplus 2} \xrightarrow{\begin{pmatrix} y & w \end{pmatrix}} A \to N \to 0 \tag{1.10}$$

where we have indicated the maps in matrix form. To compute Tor: we omit N, tensor what's left with M and take cohomology. Let's call the tensored up complex $P^\bullet$

$$P^\bullet = 0 \to M \to M^{\oplus 2} \to M \to 0 \tag{1.11}$$

By construction, $H^{-i}(P) = \mathrm{Tor}_i(M, N)$. Also by construction, $\mathrm{Tor}_0(M, N) = M \otimes N$ and we already computed its dimension to be 3. One checks that the first map $M \to M^{\oplus 2}$ is injective, as its kernel is given by those f in A/I such that $wf = 0$ and $yf = 0$ (a change of variables $x' = x + y$ etc can be helpful). This implies two things: $\mathrm{Tor}_2(M, N) = 0$ and the image $M \to M^{\oplus 2}$ is the span of the vector $(w, -y)$.

Finally, we need to compute $H^{-1}(P)$, which is given by the kernel $K = \ker(M^{\oplus 2} \to M)$ modded out by the image $I = \mathrm{im}(M \to M^{\oplus 2})$. We already know that I is the span of $(w, -y)$, so we are left with working out K. This is given by those vectors (f, g) such that $yf + wg = 0$ in M. A computation (or plugging everything into macaulay2) yields that K is generated by the obvious vector $(w, -y)$ and $(z, -x)$. One then checks that $H^{-1}P = K/I$ has dimension 1, as wanted. We will see later on that $\mathrm{Tor}_i(M, N) = 0$ for $i \geq 2$, and that $\dim_{\mathbf{C}} = \mathrm{Tor}_1(M, N) = 1$. In other words, Serre's formula gives $3 - 1 = 2$, which is the correct intersection multiplicity!

In conclusion, one finds $\chi(M, N) = 2$, as wanted. To rephrase things, the intersection $Y \cap Z$ produces the desired output only when viewed in the derived world.

1.3. Spectral sequences. — Derived categories were originally defined to perform homological algebra. For example, deriving the composition of two functors leads to a spectral sequence. Spectral sequences are a notoriously difficult subject (the concept of a spectral sequence is in itself quite elementary but the mess of indices makes it impossibly confusing). Probably the simplest case is the Leray spectral sequence. If $f \colon X \to Y$ is a map between manifolds, the spectral sequence roughly says that the cohomology of X can be computed in terms of the cohomology of Y with coefficients the cohomology groups of the fibres of f. Let us be more precise.

Let F be a sheaf (say of abelian groups) on X. The idea is that a sheaf F consists of stalks F_x for each $x \in X$, glued in a compatible way. The easiest case is $F = \mathbf{Z}_X$ the constant sheaf, which has stalk $F_x = \mathbf{Z}$ for each x and no interesting gluing information. Sheaf cohomology says $H^i(X, \mathbf{Z}_X) = H^i_{\mathrm{Betti}}(X, \mathbf{Z})$ where the latter is ordinary singular cohomology. We can push sheaves forwards, in symbols: to F we can associate its *pushforward* (or *direct image*) $f_* F$. The idea is that the stalk $(f_* F)_y$ is the space of sections $\Gamma(X_y, F|X_y)$ of F on the fibre $X_y = f^{-1}(y)$. The derived functors of f_* are the *higher direct images* $R^i f_* F$, which are sheaves on Y. The idea is that the stalk $(R^i f_* F)_y$ is the cohomology $H^i(X_y, F|X_y)$ of the fibre. The Leray spectral sequence

$$H^p(Y, R^q f_* F) \Rightarrow H^{p+q}(X, F) \tag{1.12}$$

is a way to compute the cohomology of X with coefficients in F, from the cohomology Y with coefficients in $R^i f_* F$. This is a generalization of the famous Serre spectral sequence, which can sometimes be applied to compute homotopy groups.

There is also a relative version the spectral sequence. Say $g \colon Y \to Z$ is another map, then this fancier Leray says

$$R^p g_*(R^q f_* F) \Rightarrow R^{p+q}(g \circ f)_* F \tag{1.13}$$

Indeed, the first spectral sequence is special case of the second, where $Z = \mathtt{pt}$ is a point. This is because, for $g \colon Y \to \mathtt{pt}$ the map to a point, $g_* F$ can be identified with the space of global sections $\Gamma(Y, F)$. Therefore, the higher direct images of g are the derived functors of Γ, i.e. sheaf cohomology.

Using derived categories, we replace the sequence of derived functors $R^p f_*$ with a single Rf_*. For F a sheaf, Rf_*F is a chain complex of sheaves, whose cohomology (in the sense of chain complexes: i.e. the kernel of the differential modulo the image of the differential, not be confused with *sheaf cohomology*) is $H^p(Rf_*F) = R^p f_*F$, the p-th derived functor of f_*.

Leray's spectral sequence now becomes the fact that $R(g \circ f)_*(F) = Rg_* \circ Rf_*(F)$. This formula not only looks nicer, but becomes useful when composing more than two derived functors. For example if we now wanted to take cohomology of all the sheaves involved.

Caveat emptor. Needless to say, nothing in life comes for free. Derived categories provide a great language to organize derived functors, but computations often still require spectral sequences.

2. Analogies

We continue our preliminary (and informal) discussion on derived categories. This short section has two main goals: convincing ourselves that chain complexes contain more information than their homology, understanding why derived categories would be a sensible thing to study. To achieve this, we will try and import a tiny little bit of homotopy theory in the realm of chain complexes. Once again, the details right now are not important.

2.1. Chains. — Homology is great: it takes a space X and spits out abelian groups $H_i(X)$, which interact with topological operations surprisingly well. These groups are computable in many concrete examples and, moreover, contain a bunch of information.

Theorem 2.1 **(Whitehead+Hurewicz)**. — Let X and Y be two simply-connected spaces. Then X and Y are homotopy equivalent if and only if there is a map $f: X \to Y$ inducing isomorphisms in homology $f_*: H_i(X) \to H_i(Y)$ for all i.

If you've seen this before, you know the standard warning.

Warning 2.2. — Consider $X = S^2 \vee S^4$ and $Y = \mathbf{CP}^2$. Recall that $H_i(\mathbf{CP}^2) = \mathbf{Z}$ if i is even and zero otherwise (this can easily be seen using cellular homology). Using Mayer-Vietoris, one shows that the same holds for X. Both spaces are simply-connected. However, $\pi_3(X) = \pi_3(S^2) = \mathbf{Z}$ and $\pi_3(Y) = \pi_3(S^5) = 0$, hence X and Y cannot be homotopy equivalent.

Another reason why the disembodied homology groups $H_i(-)$ do not contain enough information is the dual theory: cohomology. Consider $X = \mathbf{RP}^2$. For simplicity, let us use cellular homology. As a cell complex, X has one 0-cell e_0, one 1-cell e_1 and one 2-cell e_2. The cellular chain complex $C_\bullet^{CW}(X)$ is $\mathbf{Z}e_0$ in degree zero, $\mathbf{Z}e_1$ in degree one and $\mathbf{Z}e_2$ in degree two. The differential $C_1(X) \to C_0(X)$ sends $e_1 \mapsto 0$, the differential $C_2(X) \to C_1(X)$ sends $e_2 \mapsto 2e_1$. In other words, $C_\bullet(X)$ is given by

$$\cdots \to 0 \to \mathbf{Z} \xrightarrow{\cdot 2} \mathbf{Z} \xrightarrow{0} \mathbf{Z} \tag{2.1}$$

where the rightmost $\mathbf{Z}$ sits in homological degree zero. By definition, the homology of X is the homology of that chain complex and thus

$$\begin{cases} H_0(X) = \mathbf{Z} \\ H_1(X) = \mathbf{Z}/2\mathbf{Z} \\ H_k(X) = 0, \text{ for } k \geq 2. \end{cases} \tag{2.2}$$

Given an abelian group M, define $M^\vee = \mathrm{Hom}_{\mathbf{Z}}(M, \mathbf{Z})$. By dualizing homology we get

$$\begin{cases} H_0(X)^\vee = \mathbf{Z} \\ H_1(X)^\vee = 0 \text{ for } k \geq 1. \end{cases} \tag{2.3}$$

On the other hand, the cellular cochain complex is obtained by dualizing $C_\bullet(X)$, i.e.

$$\mathbf{Z} \xrightarrow{0} \mathbf{Z} \xrightarrow{\cdot 2} \mathbf{Z} \to 0 \to \cdots \tag{2.4}$$

where the leftmost $\mathbf{Z}$ sits in cohomological degree zero. Hence,

$$\begin{cases} H^0(X) = \mathbf{Z} \\ H^1(X) = 0 \\ H^2(X) = \mathbf{Z}/2\mathbf{Z} \end{cases} \tag{2.5}$$

From now on our motto will be:

Homology is awesome, chain complexes are awesomer.

2.2. Quasi-isomorphisms. — Let us switch to the domain of simplicial complexes, which is where the link between homotopy and chains is more direct. Given a simplicial complex X, let's write $C_\bullet(X)$ for the complex of simplicial chains. If $f\colon X \to Y$ is a simplicial map, it induces a chain map $f_*\colon C_\bullet(X) \to C_\bullet(Y)$. Theorem 2.1 says that, if X and Y are simply-connected, f is a homotopy equivalence if and only if f_* induces an isomorphism on homology. These chain maps are special and deserve a name.

Definition 2.3. — Let E, F be chain complexes. A *quasi-isomorphism* is a chain map $E \to F$ inducing an isomorphism $H_i(E) \to H_i(F)$ for all i.

This leads to another slogan.

The analogue of studying simplicial complexes up to homotopy is studying chain complexes up to quasi-isomorphism.

Of course, not every map $f\colon X \to Y$ is simplicial and so will fail to induce a chain map $C_\bullet(X) \to C_\bullet(Y)$. However, simplicial approximation comes to the rescue.

***Theorem 2.4* (simplicial approximation).** — Let $X \to Y$ be a map between simplicial complexes. There exists a third simplicial complex Z (a barycentric subdivision of X) and *simplicial* maps $q\colon Z \to X$, $g\colon Z \to Y$, such that q is a homotopy equivalence and $g = fq$.

Succinctly, at the level of spaces we have the diagram

$$\begin{array}{ccc} & Z & \\ {}^{q}\swarrow & & \searrow^{g} \\ X & \xrightarrow{\quad f \quad} & Y \end{array}$$

where the oblique maps are simplicial. At the level of complexes (of simplicial chains), while the horizontal map does not exist, the roof still makes sense.

$$\begin{array}{ccc} & C_\bullet(Z) & \\ {}^{\text{qis}}\swarrow & & \searrow \\ C_\bullet(X) & & C_\bullet(Y) \end{array}$$

This is essentially the definition of the derived category: objects are chain complexes; morphisms between two complexes E, F are given by (appropriate equivalence classes of) roofs

$$\begin{array}{ccc} & P & \\ {}^{\text{qis}}\swarrow & & \searrow \\ E & & F \end{array}$$

where $P \to E$ is a quasi-isomorphism.

3. Preliminaries

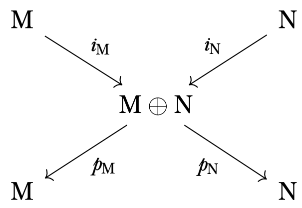
This section is a bit boring and serves as a warmup. The point is that most constructions we are familiar with in the case of modules can be carried out in a more general and abstract framework.

3.1. Modules. — Let R be a ring which, often simply due to laziness, will be assumed to be commutative. Let $\mathtt{C} = \mathtt{Mod}(R)$ be the category of (say, right) R-modules.

Remark 3.1. — Recall that a category $\mathtt{C}$ consists of objects $M, N, \ldots$ and morphisms $f : M \to N$ between objects. Composition of morphisms is associative. Every object comes with an identity $\mathrm{id}_M : M \to M$, which acts as a unit for composition. The set of morphisms between two objects M, N is denoted in various ways, such as $\mathtt{C}(M, N)$ or $\mathrm{Hom}_{\mathtt{C}}(M, N)$ or $\mathrm{Mor}_{\mathtt{C}}(M, N)$ or $[M, N]_{\mathtt{C}}$ and probably many others.

The category $\mathtt{C} = \mathtt{Mod}(R)$ has a very rich structure: it is the prototype of an *abelian* category. This means the following things.

1. The sets $\mathtt{C}(M, N) = \mathrm{Hom}_R(M, N)$ have the structure of an abelian group, which is bilinear with respect to composition of morphisms.
   - In particular, $\mathtt{C}(M, N)$ has a distinguished element: the zero morphism $0_{M,N}$ (or simply 0).
   - The group $\mathtt{C}(M, M) = \mathrm{End}_R(M)$ has two distinguished elements: id_M and 0_M.
2. There is a *zero object* 0, such that $\mathtt{C}(0, M) = 0 = \mathtt{C}(M, 0)$ for any object $M \in \mathtt{C}$.
   - An object M is the zero object if and only if $\mathrm{id}_M = 0_M$.
   - If $M, N \in \mathtt{C}$ are two objects, the zero morphism $0_{M,N}$ is the composition $M \to 0 \to N$.
3. Given two modules M, N we can form their direct sum $M \oplus N$. This can be characterized as an object $M \oplus N \in \mathtt{C}$ equipped with four maps



satisfying the following equations.

$$p_M i_M = \mathrm{id}_M \tag{3.1}$$

$$p_N i_N = \mathrm{id}_N \tag{3.2}$$

$$p_N i_M = 0 \tag{3.3}$$

$$p_M i_N = 0 \tag{3.4}$$

$$i_N p_N + i_M p_M = \mathrm{id}_{M\oplus N} \tag{3.5}$$

- As a consequence, we have $\mathrm{Hom}_R(Z, M \oplus N) = \mathrm{Hom}_R(Z, M) \oplus \mathrm{Hom}_R(Z, N)$ and $\mathrm{Hom}_R(M \oplus N, Z) = \mathrm{Hom}_R(M, Z) \oplus \mathrm{Hom}_R(N, Z)$. Categorically, this is saying that $M \oplus N$ is both a product and a coproduct of the objects M, N.

4. Given a morphism $f : M \to N$ we can form the kernel $\ker f$. This can be characterized as the fibre product $M \otimes_N 0$.
5. Dually, we have the cokernel $\mathrm{coker} f$. This can be characterized as the pushout $N \amalg_M 0$.
6. Given kernels and cokernels, we can define the *image* of a morphism: $\mathrm{im} f = \ker(N \to \mathrm{coker} f)$.
7. The first isomorphism theorem holds. More precisely, call $M/\ker f = \mathrm{coker}(\ker f \to M)$. There is a natural map $M/\ker f \to \mathrm{im} f$. This map is an isomorphism.

Remark 3.2. — What's a fibre product and a pushout? At some point I should write this up. In the specific case of Mod(R) this is easy to do.

Let $p: M \to P$ and $q: N \to P$ be two morphisms. Then their fibre product $M \times_P N$ is defined as the submodule of $M \oplus N$ given by those (x, y) such that $p(x) = q(y)$.

Let $i: P \to M$, $j: P \to N$ be two morphisms. Then their pushout $M \amalg_P N$ is defined as the module $M \oplus N$ quotiented out by the submodule given $(i(z), -j(z))$.

These two constructions satisfy a universal property.

Digression 3.3. — The object $M/\ker f = \mathrm{coker}(\ker f \to M)$ is sometimes called the *coimage* $\mathrm{coim} f$. The first isomorphism says that the natural map $\mathrm{coim} f \to \mathrm{im} f$ is an isomorphism (in particular, the notion of coimage is unnecessary).

Definition 3.4. — Let C be a category. We say C is *additive* if it satisfies (1)–(3) above, *abelian* if it satisfies (1)–(7).

In other words, an abelian category behaves very much like Mod(R): the sets C(M, N) are abelian groups, there is a zero object, direct sums, kernels, cokernels and the first isomorphism theorem holds. Here are some examples.

Example 3.5. — If R is a noetherian ring (which will be the case for all rings considered in these lectures) we write $\mathtt{Coh}(R) \subset \mathtt{Mod}(R)$ for the subcategory of finitely generated R-modules. This is an abelian category.

Example 3.6. — Let X be a variety. The category $\mathtt{Coh}(X)$ of coherent sheaves on it is abelian.

Example 3.7. — The category of projective R-modules is additive but (typically) not abelian. Why?

Example 3.8. — Let G be a discrete group. The category of finite dimensional complex representations $\mathtt{Rep}_{\mathbf{C}}(G)$ is an abelian category. It also coincides with $\mathtt{Coh}(BG)$ where BG is the quotient stack $[\mathtt{pt}/G]$.

Example 3.9. — Let Q be a quiver (aka a direct graph), which consists of a set of vertices I and a set arrows E together with maps $s, t \colon E \to I$ sending an arrow to its source vertex and target vertex. A representation of Q is the assignment of a $\mathbf{C}$-vector space V_i for each $i \in I$ and a linear map $e \colon V_{s(e)} \to V_{t(e)}$ for each arrow $e \in E$. Let's see some examples.

- The trivial quiver Q_1 consists of one vertex and no arrows. Obviously $\mathtt{Rep}Q_1 = \mathtt{Mod}(\mathbf{C})$, the category of $\mathbf{C}$-vector spaces (or the category of sheaves on a point).
- The Jordan quiver Q_2 consists of one vertex and one loop. We have $\mathtt{Rep}Q_2 = \mathtt{Mod}(\mathbf{C}[x])$.
- The Kronecker quiver Q_3 has two vertices v_1, v_2 and two arrows $v_1 \to v_2$. One can show that $D(\mathtt{Rep}Q_3) = D(\mathbf{P}^1)$, while $\mathtt{Coh}(\mathbf{P}^1) \neq \mathtt{Rep}Q_3$.

Example 3.10. — The category of Banach spaces and bounded linear maps is additive. It also admits kernels and cokernels. Indeed, if $f \colon X \to Y$ then $\operatorname{coker} f = Y/\overline{f(X)}$ is the quotient of Y module the closure of the set-theoretic image of f. However, it is *not* abelian. The reason is that the first isomorphism theorem does not hold. Take $X = C^0([0,1])$ the space of continuous $\mathbf{R}$-valued functions on the compact unit interval. Take $Y = L^1([0,1])$. The inclusion $X \to Y$ is linear and continuous and has trivial kernel. The set-theoretic image is however dense in Y, therefore the inclusion also has trivial cokernel. But $X \to Y$ is very far from being an isomorphism.

Example 3.11. — Another good non-example is the category of filtered vector spaces. Why?

Digression 3.12. — The categories above are examples of *quasi-abelian* categories, which sit in between abelian and additive ones. Depending on what you do, you will never care about these. But filtered vector spaces are most definitely an important concept.

The categories we are interested in will also have an additional piece of structure, coming from the fact that we are working over a fixed field. Indeed, if R is an algebra over $\mathbf{C}$, the set $\operatorname{Hom}_R(M, N)$ is not just an abelian group, but also a $\mathbf{C}$-vector space. We call these categories $\mathbf{C}$-linear.

Remark 3.13. — If R is a commutative $\mathbf{C}$-algebra, the set $\operatorname{Hom}_R(M, N)$ is not just a $\mathbf{C}$-vector space, but actually an R-module! However, in contrast to the things we listed above, in general there is no way to view $\mathtt{C}(M, N)$ as an object of $\mathtt{C}$. A cheap example: take R non-commutative, then $\mathrm{Hom}_R(M,N)$ cannot always be equipped with an R-module structure. A better example: take $\mathtt{C} = \mathtt{Coh}(X)$.

For R a commutative ring, when we wish to emphasize that $\mathrm{Hom}_R(M,N)$ is viewed as an object of $\mathtt{Mod}(R)$ we write $\underline{\mathrm{Hom}}_R(M,N)$. This is sometimes called the *inner hom object*.

From now on we will freely use the language of abelian categories. The way to survive is to simply pretend they are sub-categories of modules over a ring.

Digression 3.14. — There's also rigorous ways of dealing with this. Here are two. The first is to invoke the Freyd-Mitchell embedding theorem, which essentially says that if you have a diagram in some abelian category, there exist some crazy ring R and a funky way to embed that diagram in $\mathtt{Mod}(R)$. This is psychologically reassuring, as one can *literally* treat objects as modules (but one has to be careful about the precise statement of the theorem, to avoid silly mistakes). The second, is to use the Yoneda lemma. This is a more elementary approach and has the big advantage of not relying on a big theorem. It is for example explained in Aluffi's book Algebra Chapter 0.

Definition 3.15. — Let $\mathtt{C}, \mathtt{D}$ be categories. A functor $F\colon \mathtt{C} \to \mathtt{D}$ assigns to each object $M \in \mathtt{C}$ an object $F(M) \in \mathtt{D}$ together with maps $\mathtt{C}(M,N) \to \mathtt{D}(F(M),F(N))$, taking $f\colon M \to N$ to $F(f)\colon F(M) \to F(N)$. All this data satisfies: $F(\mathrm{id}_M) = \mathrm{id}_{F(M)}$ and $F(g \circ f) = F(g) \circ F(f)$.

Definition 3.16. — Let $\mathtt{C} \to \mathtt{D}$ be a functor between additive categories. We say F is additive if, for all M, N, the induced map $\mathtt{C}(M,N) \to \mathtt{D}(F(M),F(N))$ is a group homomorphism.

Example 3.17. — Let $M \in \mathtt{Mod}(R)$ be a module. Then $F(N) = M \otimes_R N$ defines a functor from $\mathtt{Mod}(R)$ to itself. This functor is additive.

Example 3.18. — Consider instead the assignment $F(M) = \bigwedge^k M$. This defines a functor $F\colon \mathtt{Mod}(R) \to \mathtt{Mod}(R)$. This functor is typically *not* additive. Why?

We only need to know two things about additive functors.

Proposition 3.19. — Additive functors preserve zeros and direct sums. This means $F(0) = 0$ and $F(M \oplus N) = F(M) \oplus F(N)$.

Proof. — Let $M \in \mathtt{C}$ be any object. Since F is a functor, $F(\mathrm{id}_M) = \mathrm{id}_{F(M)}$. Since F is additive, $F(0_M) = 0_{F(M)}$. As we mentioned earlier, we can recognize whether an object M is zero by checking $0_M = \mathrm{id}_M$. For direct sums, we use the characterization in terms of inclusions i and projections p given above, which are all preserved under an additive functor. □

3.2. Exactness. — In an abelian category, such as $\mathtt{Mod}(R)$, we can talk about exact sequences, let us recall what these are all about.

Definition 3.20. — The sequence $M \xrightarrow{f} N \xrightarrow{g} P$ is *exact* if

- $g \circ f = 0$
- the natural map $\operatorname{im} g \to \ker f$ is an isomorphism.

In general a sequence of morphisms $\cdots \to M_i \to M_{i+1} \to \cdots$ is exact if, for all i, $M_{i-1} \to M_i \to M_{i+1}$ is exact.

Remark 3.21. — Notice two things.

- $g \circ f = 0$ implies $\operatorname{im} g \subset \ker f$
- to ensure exactness, we are requiring $\operatorname{coker}(\operatorname{im} f \to \ker f) = 0$.

Exercise 3.22. — A sequence $0 \to M \to N$ is exact iff $M \to N$ is injective (ie $\ker f = 0$).
A sequence $M \to N \to 0$ is exact iff $M \to N$ is surjective (ie $\operatorname{coker} f = 0$).

Definition 3.23. — A *short exact sequence* is an exact sequence of the form

$$0 \to A \to B \to C \to 0 \tag{3.6}$$

which means $A \to B$ is injective, $B \to C$ is surjective and $\operatorname{im}(A \to B) = \ker(B \to C)$.

Proposition 3.24. — Any morphism $f\colon M \to N$ gives rise to the two exact sequences

$$0 \to \ker f \to M \to \operatorname{im} f \to 0 \tag{3.7}$$

$$0 \to \operatorname{im} f \to N \to \operatorname{coker} f \to 0 \tag{3.8}$$

Definition 3.25. — A functor $F\colon \mathtt{A} \to \mathtt{B}$ between abelian categories is *exact* if, for all short exact sequence $0 \to A \to B \to C \to 0$ in $\mathtt{A}$, the sequence $0 \to F(A) \to F(B) \to F(C) \to 0$ is short exact in $\mathtt{B}$.

4. Complexes

In this section we will finally see our protagonist: the derived category. However, we will have to wait a little before we understand how to work with it. Let us fix an arbitrary additive category $\mathtt{C}$. To fix ideas, we can pretend $\mathtt{C} = \mathtt{Mod}(R)$ or, for example, its subcategory of projective modules.

Definition 4.1. — A *complex* $(E^\bullet, d^\bullet_{E^\bullet})$ in $\mathtt{C}$ is a collection $\{E^p\}_{p\in\mathbf{Z}}$ of objects $E^p \in \mathtt{C}$ and morphisms $d^p_{E^\bullet}\colon E^p \to E^{p+1}$ such that $d^{p+1}_{E^\bullet} \circ d^p_{E^\bullet} = 0$.

Remark 4.2. — The object E^p is sometimes called the *p-chains* (or co-chains, since we are using cohomological indexing). The morphisms $d^p\colon E^p \to E^{p+1}$ are called the *differentials* (or sometimes *boundaries*, but usually when one uses homological indexing).

The best way is to represent a complex is diagrammatically as a sequence

$$\cdots \to E^{p-1} \xrightarrow{d} E^p \xrightarrow{d} E^{p+1} \to \cdots \tag{4.1}$$

where, to avoid going nuts, we omit all subscripts and superscripts from the differentials. Abusing notation we say that the sequence above is a complex if $d^2 = d \circ d = 0$. Notationally, we will also drop the $^\bullet$ superscript and simply say "E is a complex."

Remark 4.3. — By reindexing, we can pass from "cohomological" to "homological" notation. Explicitly, given a (cochain) complex $E^\bullet$ we can form a (chain) complex $E_\bullet$ by defining $E_i = E^{-i}$. The differential $E_i \to E_{i-1}$ now will decrease the degree. The difference is purely a matter of preference (and indicates whether one was brought up as a topologist or a geometer).

Definition 4.4. — A morphism between chain complexes $f\colon E \to F$ is a collection of morphisms $f^p\colon E^p \to F^p$ such that $d \circ f = f \circ d$.

In other words, the following diagram commutes

$$\begin{array}{ccc} E^p & \xrightarrow{d} & E^{p+1} \\ \downarrow f & & \downarrow f \\ F^p & \xrightarrow{d} & F^{p+1} \end{array}$$

for all $p \in \mathbf{Z}$. We write $\mathtt{Ch(C)}$ for the category of chain complexes in $\mathtt{C}$.

Proposition 4.5. — If $\mathtt{C}$ is abelian, the category $\mathtt{Ch(C)}$ is also abelian.

Proof. — Sketch the proof as an exercise. Hint: kernels and cokernels are computed levelwise, e.g. $(\ker f)^p = \ker(f^p\colon E^p \to F^p)$. □

4.1. Homology. — Homology is the real hero of these notes. Let's assume from now on that $\mathtt{C}$ is abelian.

Definition 4.6. — Let E be a complex. Define $Z^p = \ker(d\colon E^p \to E^{p+1})$ and $B^p = \operatorname{im}(d\colon E^{p-1} \to E^p)$. The p-th *cohomology* of E is the object

$$H^p(E) = Z^p/B^p = \operatorname{coker}(B^p \to Z^p) = \frac{\ker(E^p \to E^{p+1})}{\operatorname{im}(E^{p-1} \to E^p)} \tag{4.2}$$

Remark 4.7. — The object Z^p is called the p-cycles (or rather, cocycles), while B^p the p-th *boundaries.* We sometimes refer to $H^p(E)$ just as the homology of E the p-th homology instead of cohomology. After all, this homology vs cohomology business is purely bookkeeping.

Exercise 4.8. — Let E be a two-term complex, meaning $E^i = 0$ for all i except $i = 0, 1$. Call $f\colon E^0 \to E^1$. Then $H^0(E) = \ker f$, $H^1(E) = \operatorname{coker} f$.

Proposition 4.9. — Let $f\colon E \to F$ be a chain map. It induces a map $H^p(f)\colon H^p(E) \to H^p(F)$, for each p. Even better, $H^k\colon \mathtt{Ch(C)} \to \mathtt{C}$ defines an additive functor.

Proof. — Exercise. □

4.2. Quasi-isomorphisms. — The following is the key notion to define the derived category.

Definition 4.10. — A chain map $f\colon E \to F$ is a *quasi-isomorphism* if the induced map in homology $H^k(f)\colon H^k(E) \to H^k(F)$ is an isomorphism for all k.

Remark 4.11. — We sometimes abbreviate "quasi-isomorphism" to just "qis".

We are now ready for the definition of the derived category! One of our motivating examples was simplicial homology. Since we wanted to treat homotopy equivalent simplicial complexes as equal, we would have to treat quasi-isomorphic chain complexes as equal too.

***Definition 4.12* (First attempt).** — The derived category $D(\mathtt{C})$ has for objects chain complexes in $\mathtt{C}$, while the morphisms are obtained by formally declaring all quasi-isomorphisms to be isomorphisms.

The process of formally inverting morphisms is the categorical analogue of inverting elements in a ring. For example, if we take a ring R and a subset of elements $S \subset R$, its localization $R[S^{-1}]$ is defined by adding formal symbols $1/s$ for each $s \in S$. We are so used to this procedure that we feel the element $1/s$ to be a real, concrete and legitimate thing.

How would we do this at the categorical level? The category Ch(C) plays the role of R and the class of quasi-isomorphisms plays the role of $S \subset R$ above. Say $q: E \to F$ is quasi-isomorphism of chain complexes in C. This q might not be an isomorphism (just as $s \in R$ above might not have originally been invertible). The category D(C) provides a formal inverse $q' : F \to E$ (playing the role of $1/s$ in $R[S^{-1}]$).

Just as an element r could be multiplied with $1/s$ to obtain r/s, we also allow the formal inverses q' to be composable with actual morphisms in Ch(C). In other words, a morphism $E \to F$ in the derived category should be a string

$$E \overset{\text{qis}}{\leftarrow} E_1 \to E_2 \overset{\text{qis}}{\leftarrow} E_3 \to \cdots \leftarrow E_n \to F \tag{4.3}$$

which in itself is already frightening. Moreover, one should decide when two strings define the same morphism $E \to F$, which clearly leads to a combinatorial mess (not to mention set-theoretic issues). The reason this mess does not arise in localizing a ring is that one typically assumes S is a multiplicatively closed subset, which allows all elements of $R[S^{-1}]$ to be written as r/s modulo a simple equivalence relation. For this "calculus of fractions" to apply to D(C) we need to introduce first the chain homotopy category. This is why we need to wait until the next section to actually be able to work with D(C).

4.3. Digression on localizations. — There is a general procedure, called Gabriel-Zisman localization. Recall first how you localize a ring.

Definition 4.13. — Let R be a ring and let S be a multiplicatively closed subset. The localization $R[S^{-1}]$ is a ring together with a ring homomorphism $\phi: R \to R[S^{-1}]$ such that

- The element $\phi(s)$ is invertible in $R[S^{-1}]$ for all $s \in S$.
- For any other ring homomorphism $\psi: R \to A$ where, for all $s \in S$, $\psi(s)$ is invertible in A, there exists a unique ring homomorphism $\psi' : R[S^{-1}] \to A$ such that $\psi' \circ \phi = \psi$.

Let us define how to localize categories.

Definition 4.14. — Let C be any category and let S be a class of morphisms in C. The localization of C with respect to S is a category $C[S^{-1}]$ together with a functor $\phi: C \to C[S^{-1}]$ such that

- The morphism $\phi(s)$ is an isomorphism, for all arrows $s \in S$
- For any functor $\psi: C \to D$ such that, for all $s \in S$, $\psi(s)$ is an isomorphism in D, there exists a unique functor $\psi' : C[S^{-1}] \to D$ such that $\psi'\phi = \psi$.

Digression 4.15. — (a digression within a digression? this must be useless) Notice that we required the functor ψ' to be ***unique***. This implies that the localized category $C[S^{-1}]$ is characterized up to ***isomorphism***, rather than equivalence. This feels wrong (and in fact it's an example of what is called an ***evil*** concept in category theory). The correct thing to do is to require ψ' to be unique up to natural transformations. But no one cares.

As mentioned earlier, to be able to work with morphisms in the localization $\mathtt{C}[S^{-1}]$ we would need some condition on our S. Indeed, when S is a *localizing class* of morphisms, things become reasonable. Unfortunately, for quasi-isomorphisms to form a localizing class in chain complexes, we need to pass to the chain homotopy category. In any case, let's end this digression with the formal definition of the derived category.

Definition 4.16 **(Formal definition)**. — Let $\mathtt{C}$ be an abelian category. Let S be the class of all quasi-isomorphisms in $\mathtt{Ch}(\mathtt{C})$. The *derived category* $D(\mathtt{C})$ of $\mathtt{C}$ is the Gabriel-Zisman localization $\mathtt{Ch}(\mathtt{C})[S^{-1}]$.

4.4. Embedding $\mathtt{C}$ into $\mathtt{Ch}(\mathtt{C})$. — Notice that we can take an object $M \in \mathtt{C}$ and view it as a complex $M[0]$

$$\cdots \to 0 \to M \to 0 \to \cdots \tag{4.4}$$

with M sitting in degree zero. A morphism $M \to N$ extends automatically to a chain map $M[0] \to N[0]$. Thus we get a functor $\mathtt{C} \to \mathtt{Ch}(\mathtt{C})$ sending M to $M[0]$. This functor is fully faithful and exact.

Remark 4.17. — Alternatively, we could also consider $M[k]$ which views M as sitting in degree $-k$, i.e. $H^i(M[k]) = H^{i+k}(M[0])$ is zero for $i \neq -k$ and M for $i = -k$. This is compatible with the shift functor we introduce in the next section.

Proposition 4.18. — Let $F\colon \mathtt{A} \to \mathtt{B}$ be a functor between additive categories. Then F extends to a functor $\bar{F}\colon \mathtt{Ch}(\mathtt{A}) \to \mathtt{Ch}(\mathtt{B})$. In other words, upon restriction we have $\bar{F}|\mathtt{A} = F$

Proposition 4.19. — Let $F\colon \mathtt{A} \to \mathtt{B}$ be an exact functor between abelian categories. Then the extension $\bar{F}\colon \mathtt{Ch}(\mathtt{A}) \to \mathtt{Ch}(\mathtt{B})$ is also exact.

Remark 4.20. — Let M be an object of $\mathtt{C}$. From now on we will write M to indicate both the object of $\mathtt{C}$ and the object $M[0]$ of $\mathtt{Ch}(\mathtt{C})$. For F a functor as above, we will simply write F for its extension $\bar{F}$.

4.5. Snakes. — This is probably what makes the whole subject useful in the first place.

Lemma 4.21. — Let $0 \to E \to F \to G \to 0$ be a short exact sequence of chain complexes. For each k, there is a functorial map $H^{k-1}(G) \to H^k(E)$ (sometimes called the connecting morphism) such that the sequence

$$\cdots \to H^{k-1}(G) \to H^k(E) \to H^k(F) \to H^k(G) \to H^{k+1}(E) \to \cdots \tag{4.5}$$

is exact (this sequence is called the long exact sequence in homology).

The proof is a standard diagram chase once one knows the snake lemma (which is also a diagram chase). Notice that when E, F, G are two-term complexes, meaning $E^i = 0 = F^i = G^i$ for all i except for $i = 0, 1$, this is precisely the snake lemma.

Remark 4.22. — Explicitly, functoriality of the connecting morphism means that if

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E & \longrightarrow & F & \longrightarrow & G & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & E' & \longrightarrow & F' & \longrightarrow & G' & \longrightarrow & 0 \end{array}$$

is a morphism of short exact sequences of complexes (meaning that the two squares commute) then the diagram

$$\begin{array}{ccc} H^k(G) & \longrightarrow & H^{k+1}(E) \\ \downarrow & & \downarrow \\ H^k(G') & \longrightarrow & H^{k+1}(E') \end{array}$$

commutes for all k.

5. The chain homotopy category

We continue our endless discussion of homological algebra. In what follows, $\mathtt{C}$ denotes a random additive category. Sometimes, we'll also assume $\mathtt{C}$ is abelian (i.e. whenever we mention homology of a complex or exactness of a sequence).

5.1. Shift. — Let $E \in \mathtt{Ch}(\mathtt{C})$ be a chain complex. We define $E[1]$ to be E but with everything shifted one place to the left.

$$E: \qquad \cdots \xrightarrow{d} E^{p-1} \xrightarrow{d} E^p \xrightarrow{d} E^{p+1} \xrightarrow{d} \cdots$$

$$E[1]: \qquad \cdots \xrightarrow{-d} E^{p} \xrightarrow{-d} E^{p+1} \xrightarrow{-d} E^{p+2} \xrightarrow{-d} \cdots$$

The change in signs is compulsory and is just an annoying fact of life. Notice that $H^k(E[1]) = H^{k+1}(E)$.

Remark 5.1. — Let X be a topological space. The suspension ΣX of X is the space $I \times X/\sim$ where the relation identifies $(x, 0) \sim (y, 0)$ and $(x, 1) \sim (y, 1)$. The suspension isomorphism states that $\tilde{H}_{k+1}(\Sigma X) = \tilde{H}_k(X)$. This is why the *shift* $E[1]$ defined above is sometimes called the suspension.

Digression 5.2. — The analogy between shift and suspension can be made more precise but the only way I know how is by using homotopy pushouts (making this precise could be part of a presentation, for example if one wants to learn the basics of ∞-categories).

Definition 5.3. — More generally, for any integer $k \in \mathbf{Z}$ we define the *shift by k* of a complex E by

$$\begin{cases} E[k]^n = E^{n+k} \\ d^n_{E[k]} = (-1)^k d^{n+k}_E \end{cases} \tag{5.1}$$

For a chain map $f\colon E \to F$ we have $f[k]\colon E[k] \to F[k]$ defined by $f[k]^n = f^{n+k}$. Thus we have defined an exact functor $[k]\colon \mathtt{Ch}(\mathtt{C}) \to \mathtt{Ch}(\mathtt{C})$. This functor is moreover an equivalence of categories.

Notice that $[k] \circ [h] = [k + h]$.

Remark 5.4. — Look up what an equivalence of categories is.

5.2. Homotopies. — Let $f, g\colon X \to Y$ two simplicial maps between simplicial complexes. Assume that f and g are homotopic. Then, by suitably turning $I \times X$ into a simplicial complex, we find a relation between the chain maps $f_*, g_*\colon C_\bullet(X) \to C_\bullet(Y)$. This relation translates to the following.

Definition 5.5. — Let $f, g\colon E \to F$ be chain maps, i.e. morphisms in $\mathtt{Ch}(\mathtt{C})$ for some additive category $\mathtt{C}$. We say f is *homotopic* to g and write $f \sim g$ if the difference $f - g$ is null-homotopic. We say $f - g\colon E \to F$ is null-homotopic if there is a sequence $s^n\colon E^n \to F^{n-1}$ such that the following holds.

$$f^n - g^n = s^{n+1} d_E^n + d_F^{n-1} s^n \tag{5.2}$$

Proposition 5.6. — Being homotopic is an equivalence relation for $\mathtt{Ch}(\mathtt{C})$, in the sense that if $f \sim g$ then $f \circ h \sim g \circ h$ and $k \circ f \sim k \circ g$ for any composable morphisms k, h.

Definition 5.7. — Let $\mathtt{C}$ be an additive category. We define $\mathtt{K}(\mathtt{C})$, the *chain homotopy* category to have objects chain complexes in $\mathtt{C}$ and $\mathtt{K}(\mathtt{C})(E, F) = \mathtt{Ch}(\mathtt{C})(E, F)/\sim$ where $f \sim g$ if f and g are homotopic.

Unlike $\mathtt{Ch}(\mathtt{C})$, the category $\mathtt{K}(\mathtt{C})$ is (typically) *not* abelian. It is the first example of a triangulated category.

Proposition 5.8. — Homotopic maps induce the same map in homology. Concretely, let $f, g\colon E \to F$ be two chain maps. If $f \sim g$ then $H^k(f) = H^k(g)$ for all k.

Proof. — Since homology is an additive functor, it suffices to show that, for f null-homotopic, $H^k(f) = 0$ for all k. This follows from the definitions. □

Definition 5.9. — Two complexes E, F are *homotopically equivalent* if they are isomorphic in $\mathtt{K}(\mathtt{C})$. In other words there are chain maps $f\colon E \to F$, $g\colon F \to E$ such that $gf \sim \mathrm{id}_E$, $fg \sim \mathrm{id}_F$.

Remark 5.10. — Say f and g are homotopic chain maps $E \to F$. In how many ways can they be homotopic? Let's find out. Pick a homotopy s between them. Suppose t is another homotopy, then $t - s\colon E \to F[-1]$ is a legitimate chain map. Conversely, a degree minus one chain map $h\colon E \to F[-1]$ gives rise to a homotopy by taking $s + h$. In other words, the set of homotopies between f and g is a torsor for $\mathtt{Ch}(\mathtt{C})(E, F[-1])$.

5.3. Mapping Cones: topology. — The category of abelian groups (i.e. chain complexes sitting in degree zero) is the prototype of an *abelian* category: it has direct sums, kernels, cokernels, short exact sequences and the first isomorphism theorem holds. We will see that the derived category (i.e. chain complexes up to quasi-isomorphism) is the prototype of a *triangulated category.* These categories also have direct sums, but short exact sequences are replaced by *exact triangles* and kernels and cokernels are conflated into *cones.* Let's sketch a few ideas.

Let X be a simplicial complex. The *cone* over X is the topological space CX defined by taking $X \times I$ and identifying $(x, 0)$ with $(x', 0)$ for all $x, x' \in X$. This space can be given a simplicial structure as follows. We add an extra vertex t, the tip of the cone. For each vertex v, we now add a 1-simplex $[t, v]$ connecting that vertex to the tip of the cone. For each 1-simplex $[v_0, v_1]$ we add a 2-simplex $[t, v_0, v_1]$. In general, for each original k-simplex $[v_0, \dots, v_k]$ we add a $k+1$-simplex $[t, v_0, \dots, v_k]$.

Let's have a look at the complex $C_\bullet(CX)$ of simplicial chains. In degree zero we have $C_0(CX) = \mathbf{Z}t \oplus C_0(X)$, in degree one $C_1(CX) = C_0(X) \oplus C_1(X)$, where we implicitly view a 0-simplex $[v]$ as the 1-simplex $[t, v]$. In general $C_n(CX) = C_{n-1}(X) \oplus C_n(X)$. Now, given a simplex $[v_0, \dots, v_k]$, its boundary is

$$\sum_i (-1)^i [v_0, \dots, \hat{v}_i, \dots, v_k]. \tag{5.3}$$

Sticking t's everywhere has the effect of changing some signs

$$\partial_{CX}[t, v_0, \dots, v_{k-1}] = [v_0, \dots, v_{k-1}] + \sum_{i \geq 0} (-1)^{i+1} [t, v_0, \dots, \hat{v}_i, \dots, v_{k-1}]. \tag{5.4}$$

We can rewrite the boundary map in matrix form as follows.

$$\partial_{CX} = \begin{pmatrix} -\partial & 0 \\ \mathrm{id} & \partial \end{pmatrix}. \tag{5.5}$$

At this point, it's easy to check that ∂_{CX} is a differential and that $H_i(CX) = 0$ for $i > 0$.

Let now $f \colon X \to Y$ be a map. The *mapping cone* M_f is given by gluing the cone CX with Y by the rule $(x, 1) \sim f(x)$ for all $x \in X$. If f is simplicial, M_f can be given the structure of a simplicial complex. When f is moreover the inclusion of a subcomplex, this is the topologists way of taking a cokernel: since we are taking X and making it contractible (it's been coned off).

Let's have a look at the chain complexes, assuming X is a subcomplex of Y. The vertices of M_f are given by the vertices of Y plus the extra vertex t, the tip of the cone. At the level of chain complexes, this means

$$C_k(M_f) = C_{k-1}(X) \oplus C_k(Y) \tag{5.6}$$

where the boundary map sends an element in $C_{k-1}(X)$ to

$$\partial_{M_f}[t, v_0, \dots, v_{k-1}] = [f(v_0), \dots, f(v_{k-1})] - \sum_{i \geq 0} (-1)^i [t, v_0, \dots, \hat{v}_i, \dots, v_{k-1}] \tag{5.7}$$

which in matrix terms translates into

$$\partial_{M_f} = \begin{pmatrix} -\partial & 0 \\ f & \partial \end{pmatrix}. \tag{5.8}$$

Notice that f induces an inclusion $C_\bullet(X) \subset C_\bullet(Y)$ and that there is a chain map $C_\bullet(Y) \to C_\bullet(M_f)$, which sends $C_\bullet(X)$ to zero. Thus there is an induced map

$$C_\bullet(Y)/C_\bullet(X) \to C_\bullet(M_f) \tag{5.9}$$

from the quotient complex (which is in degree k is equal to $C_k(Y)/C_k(X)$). One checks that this map is a *quasi-isomorphism*. In other words, those two complexes should be considered equal in the derived category! Once we give rigorous definitions we will see mapping cones truly play the role of cokernels in the derived category.

5.4. Mapping cones: algebra. — Let now C be an additive category.

Definition 5.11. — Let $f\colon E \to F$ be a morphism in Ch(C). The *mapping cone* M_f is defined to be

$$\begin{cases} M_f^n = E^{n+1} \oplus F^n \\ d_{M_f}^n = \begin{pmatrix} -d_E^{n+1} & 0 \\ f^{n+1} & d_F^n \end{pmatrix} \end{cases} \tag{5.10}$$

In other words it is the complex $E[1] \oplus F$ with the differential twisted by f.

Notice this definition matches up with $-d_E^{n+1} = d_{E[1]}^n$. The mapping cone is sometimes called *(homotopy) cofibre* or *(homotopy) cokernel.*

Remark 5.12. — If f is zero, then $M_f = E[1] \oplus F$.

Remark 5.13. — Suppose $f\colon A \to B$ is a morphism in C, which can be viewed as a morphism of chain complexes in degree zero. The mapping cone M_f is the two term complex

$$\cdots \to 0 \to A \xrightarrow{f} B \to 0 \cdots \tag{5.11}$$

with A sitting in degree -1. We have $H^{-1}(M_f) = \ker f$, $H^0(M_f) = \operatorname{coker} f$. So M_f is in some sense playing the role of both kernel and cokernel of f.

Proposition 5.14. — If $f \sim g$ then M_f is homotopy equivalent to M_g.

Proof. — Exercise. □

5.4.1. From cones to exact sequences. — There are two obvious morphisms

$$\alpha_f \colon F \to M_f \tag{5.12}$$

$$\beta_f \colon M_f \to E[1] \tag{5.13}$$

given by projection and inclusion.

$$\alpha_f^n = \begin{pmatrix} 0 \\ \mathrm{id}_{F^n} \end{pmatrix} \tag{5.14}$$

$$\beta_f^n = \begin{pmatrix} \mathrm{id}_{E^{n+1}} & 0 \end{pmatrix} \tag{5.15}$$

Exercise 5.15. — Check α_f, β_f are indeed chain maps.

The mapping cone is a device that turns a chain map f into a short exact sequence.

Proposition 5.16. — Suppose $\mathtt{C}$ is abelian. Let $f \colon E \to F$ be a chain map. The sequence

$$0 \to F \xrightarrow{\alpha_f} M_f \xrightarrow{\beta_f} E[1] \to 0 \tag{5.16}$$

is a short exact sequence of complexes.

Proof. — Exercise. □

Remark 5.17. — Let's go back to topology for a minute. We want to see what happens when we take the cone of a cone (and then translate to chain complexes). Consider a map $f \colon X \to Y$ of topological spaces and let C_f be the mapping cone of f. There is a natural inclusion $\alpha \colon Y \to C_f$. Let C_α be its mapping cone. One can show that C_α is homotopy equivalent to the suspension ΣX.

Here is a sketch of the proof. The space C_α can be obtained by gluing the cones CX and CY by declaring $(x, 1) \sim (f(x), 1)$. For any $t < 1$, call Z_t the subspace of CX given by (x, s) for $s \leq t$. This is a subspace of CX and hence of C_α. Notice that Z_t is homotopy equivalent to CX. Now, pick $0 < t_1 < t_2 < 1$. Call W the subspace of C_α given by CY and the subspace of CX given by (x, s) with $s \geq t_2$. We can now slowly slide W down to the tip of CY. The resulting space is homotopy equivalent to the suspension of X.

Let's give the chain analogue of the remark above.

Proposition 5.18. — Let $f \colon E \to F$ be a morphism of chain complexes. We can take M_{α_f}, the mapping cone of the map $\alpha_f \colon F \to M_f$. There exists a (typically *non*-unique!) homotopy equivalence $\phi \colon E[1] \to M_{\alpha_f}$ such that the diagram

$$\begin{array}{ccccccc}
F & \xrightarrow{\alpha_f} & M_f & \xrightarrow{\beta_f} & E[1] & \xrightarrow{-f[1]} & F[1] \\
\| & & \| & & \downarrow{\scriptstyle\phi} & & \| \\
F & \xrightarrow{\alpha_f} & M_f & \xrightarrow{\alpha_{\alpha_f}} & M_{\alpha_f} & \xrightarrow{\beta_{\alpha_f}} & F[1]
\end{array}$$

commutes up to homotopy, i.e. it commutes in $\mathtt{K}(\mathtt{C})$.

Proof. — Define $\phi\colon \mathrm{E}[1] \to \mathrm{M}_{\alpha_f}$ and the inverse (up to homotopy) $\psi\colon \mathrm{M}_{\alpha_f} \to \mathrm{E}[1]$ as

$$\phi^n = \begin{pmatrix} -f^{n+1} \\ \mathrm{id} \\ 0 \end{pmatrix} \qquad \psi^n = \begin{pmatrix} 0 & \mathrm{id} & 0 \end{pmatrix}. \tag{5.17}$$

We have $\psi\phi = \mathrm{id}$, while $s^n\colon \mathrm{M}^n_{\alpha_f} \to \mathrm{M}^{n-1}_{\alpha_f}$

$$s^n = \begin{pmatrix} 0 & 0 & \mathrm{id} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.18}$$

provides a homotopy between $\phi\psi$ and id. Details can be found in [Kashiwara-Schapira, Lemma 1.4.2]. □

Remark 5.19. — The result wouldn't be true in $\mathtt{Ch}(\mathtt{C})$, the map ϕ typically does not possess an inverse on the nose, but only up to homotopy.

Warning 5.20. — This is of utmost importance (but also, depending on what you do, irrelevant in practice). The map ϕ is ***not*** unique, even up to homotopy. We will see later a criterion which guarantees uniqueness.

Lack of uniqueness can be problematic if we wish things to be sufficiently functorial. There at least two possible solutions. The first is to use a more sophisticated theory (such as dg-categories, model categories, derivators or ∞-categories) which incorporates more homotopy theory. The second, is to simply treat this as a fact of life and move on. We will opt for the latter.

Remark 5.21. — The way we rephrase the warning above is by saying that taking cones is not functorial. See the comments on the axioms below.

Proposition 5.22. — Suppose $0 \to \mathrm{E} \xrightarrow{f} \mathrm{F} \xrightarrow{g} \mathrm{G} \to 0$ is a short exact sequence of chain complexes in $\mathtt{Ch}(\mathtt{C})$. The obvious map $\mathrm{M}_f \to \mathrm{G}$ is a quasi-isomorphism and moreover $\phi \circ \alpha_f = g$.

Proof. — Exercise. □

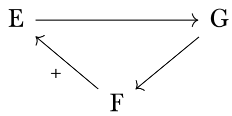
The proposition above once more confirms our intuition that mapping cones are replacing kernels/cokernels.

5.5. Triangles. — Again, consider a chain map $f\colon \mathrm{E} \to \mathrm{F}$. The sequence $\mathrm{F} \to \mathrm{M}_f \to \mathrm{E}[1]$ is the prototype of what are called exact triangles.

Definition 5.23. — We call a sequence $\mathrm{E} \to \mathrm{F} \to \mathrm{G} \to \mathrm{E}[1]$ of morphisms in $\mathtt{K}(\mathtt{C})$ a *triangle* Morphisms of triangles are what you expect (commutative diagrams in $\mathtt{K}(\mathtt{C})$). A triangle as above is called *exact* (or *distinguished*) if there is a morphism $f\colon \mathrm{E}' \to \mathrm{F}'$ and an isomorphism of triangles between $\mathrm{E} \to \mathrm{F} \to \mathrm{G} \to \mathrm{E}[1]$ and $\mathrm{E}' \to \mathrm{F}' \to \mathrm{M}_f \to \mathrm{E}'[1]$.

Once again, in the definition above commutativity of diagrams and isomorphisms are taken up to homotopy (not in $\mathtt{Ch}(\mathtt{C})$).

Remark 5.24. — Why the heck are they called triangles? Let's say anytime we write $E \overset{+}{\to} F$ we really mean a map $E \to F[1]$. Then a triangle $E \to F \to G \to E[1]$ can be drawn as



Theorem 5.25. — The collection of exact triangles in $\mathtt{K(C)}$ satisfies the following properties.

1. The class of exact triangles is closed under isomorphisms.
2. The triangle $E \overset{\mathrm{id}_E}{\to} E \to 0 \to E[1]$ is exact.
3. Given $f\colon E \to F$, there exists an exact triangle $E \overset{f}{\to} F \to G \to E[1]$.
4. The triangle $E \overset{f}{\to} F \to G \to E[1]$ is exact if and only if the triangle $F \to G \to E[1] \overset{-f[1]}{\to} F[1]$ is exact.
5. Given two exact triangles $E \overset{f}{\to} F \to G \to E[1]$, $E' \overset{f'}{\to} F' \to G' \to E'[1]$ and a commutative diagram
$$\begin{array}{ccc} E & \xrightarrow{f} & F \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} \\ E' & \xrightarrow{f'} & F' \end{array}$$
there exist a (non necessarily unique) $w\colon G \to G'$ making the obvious diagram a morphism of triangles.
6. Suppose we have two morphisms $f\colon E \to F$ and $g\colon F \to G$. Even better, suppose we have three exact triangles
$$E \overset{f}{\to} F \to C_f \to E[1]$$
$$F \overset{g}{\to} G \to C_g \to F[1]$$
$$E \overset{gf}{\to} G \to C_{gf} \to E[1].$$
there exist a distinguished triangle
$$C_f \to C_{gf} \to C_g \to C_f[1]$$
such that the following diagram is commutative

$$
\begin{array}{ccccccc}
E & \xrightarrow{f} & F & \longrightarrow & C_f & \longrightarrow & E[1] \\
\| & & \downarrow{\scriptstyle g} & & \downarrow & & \| \\
E & \xrightarrow{gf} & G & \longrightarrow & C_{gf} & \longrightarrow & E[1] \\
\downarrow{\scriptstyle f} & & \| & & \downarrow & & \downarrow{\scriptstyle f[1]} \\
F & \xrightarrow{g} & G & \longrightarrow & C_g & \longrightarrow & F[1] \\
\downarrow & & \downarrow & & \| & & \downarrow \\
C_f & \longrightarrow & C_{gf} & \longrightarrow & C_g & \longrightarrow & C_f[1]
\end{array}
$$

5.6. Comments about the axioms. —

1. This axiom is the type of stuff mathematicians live for.
2. Using the first axiom, this is equivalent to saying that, for any isomorphism $f : E \to E'$ the triangle $E \to E' \to 0 \to E[1]$ is exact. Now, cones want to be some gadget which unites both kernels and cokernels. In an abelian category, a morphism f is an isomorphism if and only if $\ker f = 0 \operatorname{coker} f$. This is what this axiom is all about.
3. Secretly, G wants to be cone(f).
4. THIS is what makes the magic of derived functors work (i.e. turning short exact sequences into long ones).
5. Again, G would like to be cone(f) and G' cone(f'). Notice once again the non-uniqueness of w. Taking cones is not a functor.
6. This last axiom is infamous (it's kind of technical) and goes by the name of *octahedral axiom*. This is because if you draw exact triangles as triangles it might look like an octahedron.

Remark 5.26. — For the last time we remark that, in order to fix the lack of functoriality of taking cones, one needs to "enhance" the category $\mathtt{K}(\mathtt{C})$ by viewing it as a dg-category or ∞-category. Using this more sophisticated framework, taking cones becomes the same as taking a homotopy pushout (respectively ∞-pushout).

5.6.1. More comments about the octahedral axiom. — Let's face it, this octahedral axiom looks kind of funky (maybe even scary). But it's actually something we want to have, because it's basically the third isomorphism theorem. Indeed, consider $M \supset N \supset P$ in an abelian category. The third isomorphism theorem says $(M/P)/(N/P) = M/N$. Label the inclusions as $f : P \to N$ and $g : N \to M$. We may rephrase the theorem as saying $\operatorname{coker}(gf)/\operatorname{coker} f = \operatorname{coker} g$ or

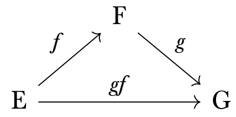
$$
\operatorname{coker}\Big(\operatorname{coker}(f) \to \operatorname{coker}(gf)\Big) = \operatorname{coker} g. \tag{5.19}
$$

Let's go back to the octahedron and recall that mapping cones want to be (homotopy) cokernels. Suppose you have a genuine chain map $f : E \to F$. Any exact triangle will be isomorphic to the standard one $E \to F \to \operatorname{cone}(f) \to E[1]$. Given another chain map $g : F \to G$ we can form $F \to G \to \operatorname{cone}(g) \to F[1]$. We can also consider the composition $gf : E \to G$ and the triangle $E \to G \to \operatorname{cone}(gf) \to E[1]$. The octahedral axiom boils down to checking that

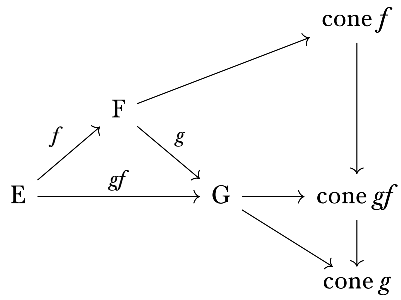
$$\text{cone}\Big(\text{cone}(f) \to \text{cone}(gf)\Big) = \text{cone}(g) \tag{5.20}$$

in other words, a “homotopy-cokernel” version of the third isomorphism theorem.

I recently learned a nice compact way of depicting the octahedral axiom (see notes by Lo who in turn cites Bayer). Given a composition



there is a commutative diagram



where the (almost) straight lines are exact.

6. Triangulated categories

We can abstract the structure of $\mathtt{K(C)}$ and define general triangulated categories.

Definition 6.1. — A *triangulated category* consists of an additive category $\mathtt{T}$ together with an autoequivalence $[1]\colon \mathtt{T} \to \mathtt{T}$ and a collection of triangles (the exact triangles) satisfying the axioms we listed above.

We typically write $[k] = [1]^k$. A triangulated functor between triangulated categories $\mathtt{T}_1 \to \mathtt{T}_2$ consists of an additive functor $F\colon \mathtt{T}_1 \to \mathtt{T}_2$ together with an isomorphism $F[1] \simeq [1]F$ such that exact triangles are sent to exact triangles.

Remark 6.2. — This is another unsatisfactory point of the theory. A triangulated category is not a category satisfying some property, it is a category plus additional structure (the collection of exact triangles). Once again, this problem goes away by considering more sophisticated theories.

Given an exact triangle $A \to B \to C \to A[1]$ we sometimes say that B is an *extension* if C by A.

Definition 6.3. — Let $\mathtt{T}$ be triangulated and $\mathtt{A}$ be abelian. A functor $F\colon \mathtt{T} \to \mathtt{A}$ is *cohomological* if, for any exact triangle $A \to B \to C \to A[1]$, the sequence $F(A) \to F(B) \to F(C)$ is exact.

Say F is cohomological and write $F^k = F \circ [k]$. For any exact triangle $A \to B \to C \to A[1]$ we know that $B \to C \to A[1] \to B[1]$ is exact. Therefore, we get a long exact sequence

$$\cdots \to F^{k-1}(C) \to F^k(A) \to F^k(B) \to F^k(C) \to F^{k+1}(A) \to \cdots \tag{6.1}$$

Proposition 6.4. — If $A \to B \to C \to A[1]$ is exact, then the composition $A \to B \to C$ is zero.

Proof. — By the axioms, $A \to A \to 0 \to A[1]$ is exact. Thus we can fill the column

$$\begin{array}{ccccccc} A & \xrightarrow{\mathrm{id}} & A & \longrightarrow & 0 & \longrightarrow & A[1] \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \phi} & & \downarrow \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & A[1] \end{array}$$

with $\phi \colon 0 \to C$ such that $gf = \phi \circ 0 = 0$. □

Remark 6.5. — Notice that for $A \to B$ chain map, the composition $A \to B \to$ cone is zero in $\mathtt{K}$ although it's typically non-zero in $\mathtt{Ch}$.

Proposition 6.6. — Let $W \in \mathtt{T}$. Then $\mathtt{T}(W, -)$ and $\mathtt{T}(-, W)$ are cohomological.

Proof. — Let's do $\mathtt{T}(W, -)$. We want to show $\mathtt{T}(W, A) \to \mathtt{T}(W, B) \to \mathtt{T}(W, C)$ is exact. Since $A \to C$ is zero, it follows $\mathtt{T}(W, A) \to \mathtt{T}(W, B) \to \mathtt{T}(W, C)$ is zero. Hence we need to show that if $\alpha \colon W \to B$ is such that $W \to B \to C$ is zero, then there exists $W \to A$ such that $W \to A \to B$ is α.

$$\begin{array}{ccccccc} W & \longrightarrow & 0 & \longrightarrow & W[1] & \longrightarrow & W[1] \\ \downarrow{\scriptstyle \alpha} & & \downarrow & & \downarrow{\scriptstyle \phi[1]} & & \downarrow \\ B & \longrightarrow & C & \longrightarrow & A[1] & \longrightarrow & B[1] \end{array}$$

We have $\phi[1] = W[1] \to A[1] \to B[1]$, thus $\phi = \phi[1][-1]$ does the trick. □

Corollary 6.7. — Let

$$\begin{array}{ccccccc} A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \xrightarrow{\gamma} & A[1] \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \omega} & & \downarrow{\scriptstyle \phi[1]} \\ D & \xrightarrow{\delta} & E & \xrightarrow{\epsilon} & F & \xrightarrow{\zeta} & D[1] \end{array}$$

If ϕ, ψ are isomorphisms, then so is ω.

Proof. — By the Yoneda lemma, to show ω is an isomorphism it suffices to show that, for any $W \in \mathtt{T}$ the induced map $\mathtt{T}(W, C) \to \mathtt{T}(W, F)$ is an isomorphism. [If you don't know what the Yoneda lemma is, don't worry. Just take what this as a (very plausible) general fact.] Apply $\mathtt{T}(W, -)$ everywhere. We obtain a commutative diagram of abelian groups with exact rows and where three out of four vertical arrows are isomorphisms. Done. □

Corollary 6.8. — Let $A \xrightarrow{f} B \to C \to A[1]$ be an exact triangle. Then f is an isomorphism if and only if $C = 0$.

Proof. — We first notice we have a morphism between exact triangles

$$\begin{array}{ccccccc} A & \xrightarrow{\mathrm{id}} & A & \longrightarrow & 0 & \longrightarrow & A[1] \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle f} & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ A & \xrightarrow{f} & B & \longrightarrow & C & \longrightarrow & A[1] \end{array}$$

and then everything follows from the proposition above. □

Proposition 6.9. — Let C be abelian. Consider the functor $H^0 \colon K(C) \to C$. It is cohomological.

Proof. — Let $A \to B \to C \to A[1]$ be an exact triangle. By definition, it's isomorphic to $E \to F \to \mathrm{cone}(f)$ for some chain map $f \colon E \to F$. By rotating the original triangle if necessary, it suffice to show that $H^0(F) \to H^0(\mathrm{cone}(f)) \to H^0(E[1])$ is exact. But we showed above that $0 \to F \to \mathrm{cone}(f) \to E[1] \to 0$ is actually a short exact sequence of complexes. The result follows. □

Proposition 6.10. — Suppose we have a (finite, for simplicity) family of exact triangles $A_i \to B_i \to C_i \to A_i[1]$. Then the triangle

$$\bigoplus_i A_i \to \bigoplus B_i \to \bigoplus C_i \to \bigoplus_i A_i[1] \tag{6.2}$$

is also exact.

Proof. — Define $A = \bigoplus A_i$, similarly B, C. By the axioms, there exists an exact triangle

$$A \to B \to Z \to A[1] \tag{6.3}$$

We have the diagram

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1] \\ \| & & \| & & & & \| \\ A & \longrightarrow & B & \longrightarrow & Z & \longrightarrow & A[1] \end{array}$$

and we are tempted to invoke the axioms for the existence of a map $C \to Z$. However, the top triangle is (a priori) not necessarily exact (otherwise we'd be done). So we do this in steps. First, the axioms do imply the existence of morphisms

$$\begin{array}{ccccccc} A_i & \longrightarrow & B_i & \longrightarrow & C_i & \longrightarrow & A_i[1] \\ \| & & \| & & & & \| \\ A & \longrightarrow & B & \longrightarrow & Z & \longrightarrow & A[1] \end{array}$$

which can added together to produce a morphism of triangles

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1] \\ \| & & \| & & \downarrow & & \| \\ A & \longrightarrow & B & \longrightarrow & Z & \longrightarrow & A[1] \end{array}$$

Once again, we would like to use the two-out-of-three proposition, which implies $C \to Z$ is an isomorphism. However, the top triangle might not be exact, so we cannot use that result. Nevertheless, we can apply $\mathtt{T}(W, -)$ and use that it is cohomological. At that point we appeal to standard homological algebra of complexes which tells us that $\mathtt{T}(W, C) \to \mathtt{T}(W, Z)$ is an isomorphism for all W. Hence, by Yoneda, $C \to Z$ is an isomorphism. Therefore the top triangle is isomorphic to an exact one, hence it is itself exact. □

Corollary 6.11. — Consider two objects E, F then the triangle

$$E \to E \oplus F \to F \xrightarrow{0} E[1] \tag{6.4}$$

is exact.

Proof. — We have exact triangles $E \to E \to 0 \to E[1]$ and $0 \to G \to G \to 0$. Add them together and use the proposition above. □

Corollary 6.12. — Let $A \to B \to C \to A[1]$ be an exact triangle. Suppose $C \to A[1]$ is the zero morphism. Then the triangle is split, i.e. it is isomorphic to the triangle $A \to A \oplus C \to C \xrightarrow{0} A[1]$.

Corollary 6.13. — Let $A \to B \to C \to A[1]$ be an exact triangle. Suppose there is $B \to A$ such that $A \to B \to A$ is the identity. Then the triangle is split.

7. Derived categories (finally!)

Convention:— from now on I might write exact triangles as $E \to F \to G$, omitting the morphism $G \to E[1]$. It shouldn't create confusion.

We are ready for the definition of derived category. Notice that our starting point is the chain homotopy category $\mathtt{K}(\mathtt{C})$. For example, this means commutative diagrams of chain maps will actually commute only up to homotopy.

Definition 7.1. — Let $\mathtt{C}$ be an abelian category, its derived category $\mathtt{D}(\mathtt{C})$ is defined as follows. The objects are the same as those of $\mathtt{Ch}(\mathtt{C})$ and $\mathtt{K}(\mathtt{C})$: chain complexes. Morphisms $\mathtt{D}(E, F)$ are given by equivalence classes of diagrams in $\mathtt{K}(\mathtt{C})$ of the shape

$$\begin{array}{ccc} & Q & \\ {}^{\text{qis}}\swarrow & & \searrow \\ A & & B \end{array}$$

where $Q \to A$ is a quasi-isomorphism (there is also a theory with the reverse chirality, where $Q \to B$ is assumed to be a quasi-isomorphism). We declare two diagrams $A \leftarrow Q \to B$, $A \leftarrow R \to B$ to be equivalent if there exists a commutative diagram in $\mathtt{K}(\mathtt{C})$

$$\begin{array}{ccccc} & & Q & & \\ & {}^{\text{qis}}\swarrow & \uparrow & \searrow & \\ A & \xleftarrow{\text{qis}} & S & & B \\ & {}_{\text{qis}}\nwarrow & \downarrow & \nearrow & \\ & & R & & \end{array}$$

However we still haven't said how to compose morphisms, nor why $\mathtt{D}(\mathtt{C})$ should even be an additive, let alone triangulated, category. Before we do so, let's place this definition in a slightly more general context.

Definition 7.2. — A complex E is *acyclic* if $\mathrm{H}^p(\mathrm{E}) = 0$ for all $p \in \mathbf{Z}$.

Acyclic complexes form what [Kashiwara-Schapira] calls a null system, i.e.

1. 0 is acyclic
2. E is acyclic if and only if E[1] is acyclic
3. If $\mathrm{E} \to \mathrm{F} \to \mathrm{G} \to \mathrm{E}[1]$ is an exact triangle with E, G acyclic then F is acyclic.

Remark 7.3. — A morphism $f\colon \mathrm{E} \to \mathrm{F}$ in $\mathtt{K}$ is a quasi-isomorphism if and only if there is an exact triangle $\mathrm{E} \to \mathrm{F} \to \mathrm{G}$ with G acyclic.

The remark and the properties above are all obvious since H^0 is a cohomological functor.

Thus we see that forcing all quasi-isomorphisms to be zero is the same as declaring all acyclic complexes to be zero.

Proposition 7.4. — The class of quasi-isomorphisms is what [Kashiwara-Schapira] calls a (left and right) multiplicative system (which other people call a localizing class). This means that

1. id_E is a quasi-isomorphism for any E
2. if $f\colon \mathrm{E} \to \mathrm{F}$ is a qism and $g\colon \mathrm{F} \to \mathrm{G}$ is a qism then $gf\colon \mathrm{E} \to \mathrm{G}$ is a qism
3. given maps $\mathrm{E} \to \mathrm{Z} \leftarrow \mathrm{F}$, with $\mathrm{E} \to \mathrm{Z}$ a qism, there exists a commutative diagram

$$\begin{array}{ccc} \mathrm{A} & \xrightarrow{\text{qis}} & \mathrm{F} \\ \downarrow & & \downarrow \\ \mathrm{E} & \xrightarrow{\text{qis}} & \mathrm{Z} \end{array}$$

   with $\mathrm{A} \to \mathrm{F}$ a qism. Some thing but starting with $\mathrm{E} \leftarrow \mathrm{A} \to \mathrm{F}$ and completing the south-east corner of the diagram.
4. Say $f, g\colon \mathrm{E} \to \mathrm{F}$. The following are equivalent
   (a) there is a qism $q\colon \mathrm{F} \to \mathrm{Z}$ such $qf = qg$
   (b) there is a qism $t\colon \mathrm{A} \to \mathrm{E}$ such that $ft = gt$.

Proof. — 1. obvious
2. by the octahedral axiom (fun!), the sequence $\mathrm{cone}(f) \to \mathrm{cone}(gf) \to \mathrm{cone}(g)$ is exact. The first and third are zero, by assumption, hence $\mathrm{cone}(gf)$ must also be zero (for example because $\mathtt{K}(\mathrm{W}, -)$ is cohomological).
3. We know we have an exact triangle $\mathrm{E} \to \mathrm{Z} \to \mathrm{N}$ with N acyclic. By using the composition $\mathrm{F} \to \mathrm{Z} \to \mathrm{N}$ we get an exact triangle $\mathrm{F} \to \mathrm{N} \to \mathrm{W}$. The axioms yield

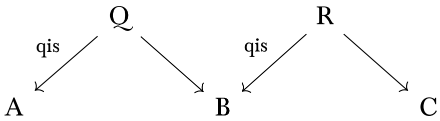
$$\begin{array}{ccccccc} \mathrm{W}[-1] & \longrightarrow & \mathrm{F} & \longrightarrow & \mathrm{N} & \longrightarrow & \mathrm{W} \\ \downarrow & & \downarrow & & \| & & \downarrow \\ \mathrm{E} & \longrightarrow & \mathrm{Z} & \longrightarrow & \mathrm{N} & \longrightarrow & \mathrm{E}[1] \end{array}$$

   By setting $\mathrm{A} = \mathrm{W}[-1]$ we are done as $\mathrm{A} \to \mathrm{F}$ must be a qism since N is acyclic. The opposite situation with arrows reversed is analogous.

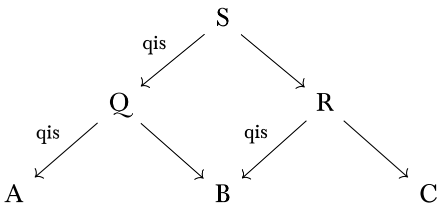
4. Omitted.

□

Using these properties we can define composition in $\mathtt{D}(\mathtt{C})$ and make sure it's well defined. Indeed, suppose we have roofs in $\mathtt{K}(\mathtt{C})$

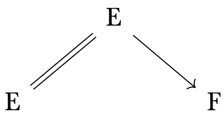


then we know there exists $Q \leftarrow S \rightarrow R$ with $S \rightarrow Q$ a qism making the diagram commute



So we define the composition $A \leftarrow Q \rightarrow B$ with $B \leftarrow R \rightarrow C$ as $A \leftarrow Q \leftarrow S \rightarrow R \rightarrow C$. Then we need to make sure it is well defined.

The derived category $\mathtt{D}(\mathtt{C})$ defined this way is the localization of $\mathtt{K}(\mathtt{C})$ along quasi-isomorphisms. This means the following. We have a functor $Q\colon \mathtt{K}(\mathtt{C}) \rightarrow \mathtt{D}(\mathtt{C})$ leaving objects fixed and taking a morphism $f\colon E \rightarrow F$ to the roof



The functor Q sends a quasi-isomorphism to an isomorphism. Moreover, for any category $\mathtt{T}$ and functor $F\colon \mathtt{K}(\mathtt{C}) \rightarrow \mathtt{T}$ which sends quasi-isomorphisms to isomorphisms, there is a unique functor $\overline{F}\colon \mathtt{D}(\mathtt{C}) \rightarrow \mathtt{T}$ such that $F = \overline{F} \circ Q$. This property characterizes $\mathtt{D}(\mathtt{C})$.

Proposition 7.5. — The category $\mathtt{D}(\mathtt{C})$ is triangulated, the obvious functor $Q\colon \mathtt{K}(\mathtt{C}) \rightarrow \mathtt{D}(\mathtt{C})$ is triangulated.

We can also characterize $\mathtt{D}(\mathtt{C})$ in the following way. The functor Q sends acyclic complexes to zero. Moreover, Q is universal with respect to functors $F\colon \mathtt{K}(\mathtt{C}) \rightarrow \mathtt{T}$ where $\mathtt{T}$ is triangulated and F sends acyclic complexes to zero.

Proof. — The only thing to say is that we declare a triangle in $\mathtt{D}(\mathtt{C})$ to be exact if it's the image of an exact triangle in $\mathtt{K}(\mathtt{C})$. The rest is just super tedious but follows directly from the axioms. □

Corollary 7.6. — The functor H^0 descends to a well defined (and cohomological) functor on $\mathtt{D}$.

Proposition 7.7. — Consider $\mathtt{K}(\mathtt{C}) \rightarrow \mathtt{D}(\mathtt{C})$. Let $E \in \mathtt{K}$. The following are equivalent.

1. E becomes zero in $\mathtt{D}$.
2. There is F such that $E \oplus F$ is acyclic.

3. E is acyclic.

Proof. — If E is zero in $\mathtt{D}$ then the roof $E \leftarrow E \xrightarrow{0} E$ must be equivalent to the identity. This means there is Q such that $Q \xrightarrow{0} E$ is a qism. Embed this in an exact triangle $Q \to E \to G \to Q[1]$. Since $Q \to E$ is a qism we have G is acyclic. Since $Q \to E$ is zero, we have $G = E \oplus Q[1]$.

Suppose now $E \oplus F$ is acyclic. Then, by definition, $H^n(E \oplus F) = H^n(E) \oplus H^n(F) = 0$ for all n. Which implies E is acyclic.

Say now E is acyclic. This means the map $E \to 0$ is a qism and hence invertible in the derived category. In other words $E = 0$ in $\mathtt{D}$. □

Proposition 7.8. — Say $f \colon E \to F$ is a morphism in $\mathtt{K}$. Then f is an isomorphism in $\mathtt{D}$ if and only if f is a qism.

Proof. — Embed f in an exact triangle $E \to F \to G \to$. Since f becomes an iso in $\mathtt{D}$, it means $G = 0$ in $\mathtt{D}$ (by general properties of triangulated categories). By the previous proposition, G is acyclic. By taking H^0 we see that f is a qism. □

Proposition 7.9. — Say $f \colon E \to F$ is a morphism in $\mathtt{K}$. Then $f = 0$ in $\mathtt{D}$ if and only if there exists a qism $Q \to E$ such that $Q \to E \to F$ is homotopic to zero if and only there is a qism $F \to R$ such that $E \to F \to R$ is homotopic to zero.

Proof. — omitted. □

Let's have a look at an example.

Example 7.10. — Consider $\mathtt{C} = \mathtt{Coh(k)}$, i.e. finite dimensional vector spaces over a field $\mathtt{k}$ (or, more generally, an abelian category where any short exact sequence splits). Let $E \in \mathtt{D(C)}$, then $E \simeq \bigoplus_i H^i(E)[-i]$.

Proof. — Let E be a chain complex of vector spaces. Call $H = \bigoplus_i H^i(E)[-i]$ the chain complex (with zero differentials) of its homology. Recall that $H^p(E) = Z^p(E)/B^p(E)$. But since we are in vector spaces, we can split all quotients, so that $E^p = H^p \oplus S^p$ for some choice of S^p. Thus we have a well defined map $E \to H$ by projecting E^p to the first factor. This map is obviously a quasi-isomorphism. □

Proposition 7.11. — Let $0 \to A \to B \to C \to 0$ be a short exact sequence in $\mathtt{Ch(C)}$. Then there exists $C \to A[1]$ in $\mathtt{D(C)}$ such that $A \to B \to C \to A[1]$ is an exact triangle.

Conversely, suppose we have an exact triangle $A \to B \to C \to A[1]$ in $\mathtt{D(C)}$ where $A, B, C \in \mathtt{C}$. Then it comes from a short exact sequence $0 \to A \to B \to C \to 0$ in $\mathtt{C}$.

Proof. — Consider $Z = \mathrm{cone}(A \to B)$. We know $A \to B \to Z \to A[1]$ is an exact triangle. We saw earlier that there is a natural qism $Z \to C$. This now admits an inverse in $\mathtt{D}$ so that we may define a triangle $A \to B \to C \to A[1]$.

For the second statement, take H^0. □

7.1. Reminder. — Before we go on, let's remind ourselves how we got here. We started with an abelian category C, for example C = Mod(R). We looked at chain complexes Ch(C). Two chain complexes are isomorphic in Ch if there are isomorphic on the nose. But this wasn't good enough, so we declared homotopic maps to be equivalent, which lead to the chain homotopy category K(C). Here, two chain complexes are the same if they are isomorphic up to homotopy. But this wasn't good enough, so we declared all quasi-isomorphisms to be invertible, which lead to D(C), the derived category. Here, all acyclic complexes become zero.

Let's have a look at two examples. The chain complex of abelian groups

$$\mathrm{E}\colon \cdots \to 0 \to \mathrm{Z} \xrightarrow{\mathrm{id}} \mathbf{Z} \to 0 \to \cdots \tag{7.1}$$

is not isomorphic to the zero complex. However, it is chain homotopy equivalent to it. Hence, E and 0 are the same in K.

On the other hand

$$\mathrm{F}\colon \cdots \to 0 \to \mathbf{Z} \xrightarrow{\cdot 2} \mathbf{Z} \to \mathbf{Z}/2\mathbf{Z} \to 0 \to \cdots \tag{7.2}$$

is neither isomorphic to zero nor homotopy equivalent to it. However, the map $\mathrm{F} \to 0$ is a quasi-isomorphism (i.e. F is acyclic) hence F and 0 are the same in D.

7.2. Truncations. — Let E be a chain complex. Fix an index p, we define

$$\tau^{\leq p}\mathrm{E}\colon \quad \cdots \to \mathrm{E}^{p-2} \to \mathrm{E}^{p-1} \to \ker(\mathrm{E}^{p} \to \mathrm{E}^{p+1}) \to 0 \to \cdots \tag{7.3}$$

$$\tau^{\geq p}\colon \quad \cdots \to 0 \to \operatorname{coker}(\mathrm{E}^{p-1} \to \mathrm{E}^{p}) \to \mathrm{E}^{p+1} \to \mathrm{E}^{p+1} \to \cdots \tag{7.4}$$

and call them *truncations* of E. Clearly, a chain map $f\colon \mathrm{E} \to \mathrm{F}$ induces a map between the truncations. One checks that $\tau\colon$ Ch(C) $\to$ Ch(C) is a functor, for whatever choice of superscript. Moreover, if f is homotopic to g then $\tau(f)$ is homotopic to $\tau(g)$. Hence τ descends to a functor $\tau\colon$ K(C) $\to$ K(C).

We have obvious maps

$$\tau^{\leq p}\mathrm{E} \to \mathrm{E} \tag{7.5}$$

$$\mathrm{E} \to \tau^{\geq p}\mathrm{E} \tag{7.6}$$

Notice

$$\mathrm{H}^{i}(\tau^{\leq p}\mathrm{E}) = \begin{cases} \mathrm{H}^{i}(\mathrm{E}) & i \leq p \\ 0 & i > p \end{cases} \tag{7.7}$$

and dually

$$\mathrm{H}^{i}(\tau^{\geq p}\mathrm{E}) = \begin{cases} 0 & i < p \\ \mathrm{H}^{i}(\mathrm{E}) & i \geq p \end{cases} \tag{7.8}$$

Hence if $\mathrm{E} \to \mathrm{F}$ is a qism, then $\tau(\mathrm{E}) \to \tau(\mathrm{F})$ will also be a qism. This means τ descends to a functor $\tau\colon$ D(C) $\to$ D(C).

Proposition 7.12. — Let E be a complex. The following triangles are exact in $\mathtt{D}$.

$$\tau^{\leq p}\mathrm{E} \to \mathrm{E} \to \tau^{>p}\mathrm{E} \to \tag{7.9}$$

$$\tau^{\leq p-1}\mathrm{E} \to \tau^{\leq p}\mathrm{E} \to \mathrm{H}^{p}(\mathrm{E})[-p] \to \tag{7.10}$$

$$\mathrm{H}^{p}(\mathrm{E})[-p] \to \tau^{\geq p}\mathrm{E} \to \tau^{\geq p+1}(\mathrm{E}) \to \tag{7.11}$$

7.3. More truncations. — We need to discuss some easy variants of $\mathtt{D}(\mathtt{C})$. We say a chain complex E is *bounded above* if $\mathrm{E}^p = 0$ for $p \gg 0$, *bounded below* $\mathrm{E}^p = 0$ for $p \ll 0$. We say E is *bounded* if it's both bounded above and below. We write $\mathtt{D}^-(\mathtt{C}), \mathtt{D}^+(\mathtt{C}), \mathtt{D}^{\mathrm{b}}(\mathtt{C})$ for the (isomorphism closures of the) subcategories of (respectively) bounded above, bounded below and bounded chain complexes. These are triangulated subcategories of $\mathtt{D}(\mathtt{C})$.

Proposition 7.13. — Let E be a chain complex. Then $\mathrm{E} \in \mathtt{D}^-(\mathtt{C})$ if and only if $\mathrm{H}^i(\mathrm{E}) = 0$ for $i \gg 0$. Similarly for the bounded below and bounded subcategories.

Proof. — Suppose $\mathrm{H}^i(\mathrm{E}) = 0$ for $i \geq p$. Then $\tau^{\leq p}\mathrm{E} \to \mathrm{E}$ is a quasi-isomorphism. □

These subcategories are important when we'll want to define derived functors.

Truncations can also be used to 'filter' a complex by its homology. Indeed, we may consider the sequence

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \tau^{\leq p-1}\mathrm{E} & & \longrightarrow & \tau^{\leq p}\mathrm{E} & & \longrightarrow & \tau^{\leq p+1}\mathrm{E} & \longrightarrow \cdots \\ & & \nwarrow_{+} & \swarrow & & \nwarrow_{+} & \swarrow & & \\ & & & \mathrm{H}^{p}(\mathrm{E})[-p] & & & \mathrm{H}^{p+1}(\mathrm{E})[-p-1] & & \end{array}$$

which we interpret as giving a filtration of E by (shifts of) objects in $\mathtt{C}$. If we end up talking about t-structures, we will make this more precise.

8. Derived functors

OK, derived categories are only useful because we have derived functors, so let's see what those are. Consider an additive functor $\mathrm{F}\colon \mathtt{A} \to \mathtt{B}$. This obviously extends to a functor $\mathtt{Ch}(\mathtt{A}) \to \mathtt{Ch}(\mathtt{B})$ and in turn to a functor $\mathtt{K}(\mathtt{A}) \to \mathtt{K}(\mathtt{B})$, which we still denote by F. By composing with $\mathtt{K}(\mathtt{B}) \to \mathtt{D}(\mathtt{B})$, we have a diagram

$$\begin{array}{ccc} \mathtt{K}(\mathtt{A}) & \longrightarrow & \mathtt{D}(\mathtt{B}) \\ {\scriptstyle \mathrm{Q}}\downarrow & & \\ \mathtt{D}(\mathtt{A}) & & \end{array}$$

which we wish to close up with some functor

$$\begin{array}{ccc} \mathtt{K}(\mathtt{A}) & \longrightarrow & \mathtt{D}(\mathtt{B}) \\ {\scriptstyle \mathrm{Q}}\downarrow & \nearrow_{\mathrm{LF}} & \\ \mathtt{D}(\mathtt{A}) & & \end{array}$$

However, requiring that $LF \circ Q = F$ is too strong of a condition. Hence, we require the next best thing: *Kan extensions.*

Definition 8.1. — The *left Kan extension* of F along Q (or maybe of Q along F) is a functor LF together with a natural transformation $LFQ \to F$ which is universal. Explicitly, for any other $G\colon \mathtt{D}(\mathtt{A}) \to \mathtt{D}(\mathtt{B})$ and transformation $LF \circ Q \to G$ there is a bijection $[G \circ Q, F] \to [LF, G]$.

In this context, LF is called the *left derived functor* of F. Dually, we can define *right derived functors.*

8.1. Something more useful. — OK, but this definition is useless. How to compute LF in practice? Let E be an object of A, or maybe even a complex. The point is that F(E) needs to be 'corrected'. We need to find a complex E', quasi-isomorphic to E, which behaves better with respect to F. Then $LF(E) = F(E')$.

Example 8.2. — Consider in $\mathtt{Mod}(R)$ the functor $F(N) = N \otimes_R M$ for a fixed M. We know that F is *right exact* in the sense that it takes a short exact sequence $0 \to N' \to N \to N'' \to 0$ to an exact sequence $F(N') \to F(N) \to F(N'') \to 0$. To fix the lack of exactness on the left, we need to derive F to the left. To achieve this, we must replace M with an improved version of it. This means finding a complex P, where each P^i is a free module, and a quasi-isomorphism between P and M. Then $LF(N) = P \otimes N$ will be our derived functor. Notice $P \otimes N$ is a chain complex, defined only up to quasi-isomorphisms, i.e. it's only well defined in $\mathtt{D}(R)$. Traditionally, one calls $\mathrm{Tor}_i(M, N) = H^{-i}(LF(N)) = H^{-i}(P \otimes N)$.

Let's see how this works in general.

Proposition 8.3. — Let A be an abelian category. Suppose P is a *generating* additive subcategory, where generating means that for any $M \in \mathtt{A}$ there is a surjection $F \twoheadrightarrow M$ with $F \in \mathtt{P}$. Then, given any complex $E \in \mathtt{K}^-(\mathtt{A})$ there exists a complex $P \in \mathtt{K}^-(\mathtt{P})$ and a quasi-isomorphism $P \to E$. Even better, the natural functor $\mathtt{K}^-(\mathtt{A})/\mathrm{acy} \to \mathtt{D}^-(\mathtt{A})$ is an equivalence.

In particular, if we wish to define LF on $\mathtt{D}^-(\mathtt{A})$ we might as well define it on $\mathtt{K}^-(\mathtt{P})$. We call the complex $P^\bullet$ quasi-isomorphic to E a *resolution* of E by objects in P.

Proof. — Consider $E \in \mathtt{K}^-$. Up to shifting, we can just assume $E^i = 0$ for $i > 0$. Also, let's switch to homological notation, for simplicity. The first thing we can do is find a surjection $P_0 \twoheadrightarrow E_0$. By composition, the map $P_0 \to E_0 \to H_0(P)$ is also surjective. Notice that we have an exact sequence

$$0 \to Z_1 \to E_1 \to E_0 \to H_0 \to 0$$

which induces an exact sequence

$$0 \to H_1 \to \frac{E_1}{B_1} \to E_0 \to H_0 \to 0.$$

We now define $F = \frac{E_1}{B_1} \times_{E_0} P_0$, $G = E \times_{\frac{E_1}{B_1}} F = E \times_{E_0} P_0$. By assumption, we can find a surjection $P_1 \twoheadrightarrow G$. Pictorially,

$$\begin{array}{ccccccc} P_1 & \twoheadrightarrow & G & \longrightarrow & F & \longrightarrow & P_0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & E_1 & \longrightarrow & \frac{E_1}{B_1} & \longrightarrow & E_0 \end{array}$$

Notice that $P_1 \to E_1 \to E_0 = P_1 \to P_0 \to E_0$. Thus we have constructed a chain map

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & E_2 & \longrightarrow & E_1 & \longrightarrow & E_0 & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}.$$

Moreover, one checks by inspection that $P_0/P_1 = H_0$ and $\ker(P_1 \to P_0) \to H_1$ is surjective.

Let us now define the next term. We know there is an exact sequence

$$0 \to H_2 \to \frac{E_2}{B_2} \to Z_1 \to H_1 \to 0$$

By construction, we have a well defined surjection $Z_1(P) \to Z_1$. Hence we can take fibre products again, with end result

$$\begin{array}{ccccccc} P_2 & \twoheadrightarrow & G' & \longrightarrow & F' & \longrightarrow & Z_1(P) \\ & & \downarrow & & \downarrow & & \downarrow \\ & & E_2 & \longrightarrow & \frac{E_2}{B_2} & \longrightarrow & Z_1 \end{array}$$

Thus, we have constructed a chain map

$$\begin{array}{ccccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & E_3 & \longrightarrow & E_2 & \longrightarrow & E_1 & \longrightarrow & E_0 & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}.$$

such that $H_i(P) \to H_i$ is an isomorphism for $i = 0, 1$ and surjective for $i = 2$. We can repeat this last step to construct the full resolution (induction). The second assertion is plausible. □

Let us mention two general facts we (not so) secretly just used in the proof above.

Lemma 8.4. — Let E be a chain complex. There is a short exact sequence

$$0 \to H^p(E) \to E^p/B^p \to Z^{p+1} \to H^{p+1} \to 0 \tag{8.1}$$

Recall, $Z^p = \ker(E^p \to E^{p+1})$, $B^p = \operatorname{im}(E^{p-1} \to E^p)$, $H^p = Z^p/B^p$.

Lemma 8.5. — Suppose $A \to C \leftarrow B$ are morphisms in an abelian category $\mathtt{A}$. Consider the fibre product

$$\begin{array}{ccc} F & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & C \end{array}$$

then $\ker(F \to B) = \ker(A \to C)$. Moreover, if $B \twoheadrightarrow C$ is surjective, then $F \twoheadrightarrow A$ is surjective and $\mathrm{coker}(F \to B) = \mathrm{coker}(A \to C)$.

Remark 8.6. — Of course, one would want something similar to hold for the unbounded derived category. This can be achieved, by using what are called K-projective (a.k.a. q-projective a.k.a. ho-injective, depends whom you ask) resolutions. But it gets rather technical.

The following proposition is psychologically comforting.

Proposition 8.7. — Suppose $\mathtt{A}$ has enough projectives and call $\mathtt{P}$ the subcategory of projective objects. Then $\mathtt{K}^-(\mathtt{P}) = \mathtt{D}^-(\mathtt{A})$.

Recall that an object P is *projective* if any short exact sequence $0 \to A \to E \to P \to 0$ splits. We say $\mathtt{A}$ has *enough projectives* if, given $A \in \mathtt{A}$, there is a surjection $P \twoheadrightarrow A$.

Proof. — We only need to show that $\mathrm{acy} \cap \mathtt{K}^-(\mathtt{P}) = 0$. In other words, if E is an acyclic bounded above complex with each E^i projective, then E is homotopy equivalent to zero (and not merely quasi-isomorphic to zero).

To ease notation, we switch to homological indexing and assume $E_i = 0$ for $i < 0$. Since E is acyclic, $B_p = Z_p$ for all p. Hence we have short exact sequences

$$0 \to Z_{p+1} \to E_{p+1} \to Z_p \to 0.$$

Let's have a look at $p = 0$. Since $Z_0 = E_0$, which is projective, we can pick a splitting $E_1 = Z_0 \oplus Z_1$. Since the direct summand of a projective object is also projective, it follows Z_1 is projective. Hence, $E_2 = Z_1 \oplus Z_2$ and thus Z_2 is projective. By induction, we may pick splittings of all those exact sequences. In turn, we may define a homotopy between the identity of E and zero by using the splittings $E_p \to Z_p \to E_{p+1}$. □

Digression 8.8. — This is actually a general feature of (well behaved) localizations. Any time you have a category $\mathtt{C}$ and take a localization $\mathtt{C}[S^{-1}]$ you should expect to find a subcategory $\mathtt{C}_S \subset \mathtt{C}$ which is equivalent to the localization. This happens for example with sheaves. Indeed, take $\mathtt{C}$ to be the category of presheaves on a topological space (or site). Typically one defines sheaves as a special class of presheaves. But one could also localize. Indeed, declare a morphism $E \to F$ of presheaves to be a local-isomorphism of it induces an isomorphism on stalks. Then the localization of $\mathtt{C}$ at local-isomorphisms is equivalent to the category of sheaves.

Digression 8.9. — Notice that in the digression above one needs to be careful not to fall into set-theoretic traps. For example, it is well known that there is no fpqc sheafification of a presheaf. Therefore the localization of presheaves with respect to fpqc-local-isomorphisms is an ill-defined category (in a sense one can make precise). This is very much like Russell's set of all sets.

OK, how to we define derived functors then?

Definition 8.10. — Suppose $F\colon \mathtt{A} \to \mathtt{B}$ is a right exact functor between abelian categories. An additive subcategory $\mathtt{P} \subset \mathtt{A}$ is called F*-projective* if

1. For any $M \in \mathtt{A}$, there exists there exists $P \in \mathtt{P}$ and a surjection $P \twoheadrightarrow M$

2. if $0 \to M \to P \to P'' \to 0$ is short exact and P, P'' are in $\mathtt{P}$ then M is also in $\mathtt{P}$.
3. if $0 \to P' \to P \to P'' \to 0$ is short exact in $\mathtt{A}$ with P', P, P'' in $\mathtt{P}$, then $0 \to F(P') \to F(P) \to F(P'') \to 0$ is short exact in $\mathtt{B}$.

Let's have a look at the composition $\mathtt{K}^-(\mathtt{A}) \to \mathtt{K}^-(\mathtt{B}) \to \mathtt{D}^-(\mathtt{B})$. We want to define $\mathtt{D}^-(\mathtt{A}) \to \mathtt{D}^-(\mathtt{B})$. We know that $\mathtt{D}^-(\mathtt{A}) = \mathtt{K}^-(\mathtt{P})/\text{acy}$. Thus, it suffices to show the following.

Proposition 8.11. — With assumptions and notation as above, if $P \in \mathtt{K}^-(\mathtt{P})$ is acyclic, then $F(P) = 0$ in $\mathtt{D}^-(\mathtt{B})$.

Proof. — Suppose $P \in \mathtt{K}^-(\mathtt{P})$ is acyclic and (once more) let's use homological indexing and assume $P_i = 0$ for i negative. Then we have a short exact sequence

$$0 \to Z_1 \to P_1 \to P_0 \to 0$$

By (2), $Z_1 \in \mathtt{P}$ and by (3)

$$0 \to F(Z_1) \to F(P_1) \to F(P_0) \to 0$$

is still exact. We proceed by induction and conclude. □

Remark 8.12. — This whole section works also backwards: with left exact functors, F-*injective* categories and the bounded below derived category.

Definition 8.13. — Let $F\colon \mathtt{A} \to \mathtt{B}$ be a left exact functor between abelian categories. Suppose $RF\colon \mathtt{D}^+(\mathtt{A}) \to \mathtt{D}^+(\mathtt{B})$ exists. We say an object $M \in \mathtt{A}$ is F-*acyclic* if $R^iF(M) = 0$ for $i \neq 0$.

Proposition 8.14. — Suppose $F\colon \mathtt{A} \to \mathtt{B}$ is an explicitly right derivable left exact functor between abelian categories with right derived functor $RF\colon \mathtt{D}^+(\mathtt{A}) \to \mathtt{D}^+(\mathtt{B})$. Suppose we have an exact sequence

$$0 \to M \to C^0 \to C^1 \to C^2 \to \cdots$$

where the C^i are F-acyclic. We can form the complex

$$F(C^\bullet) = \cdots \to 0 \to F(C^0) \to F(C^1) \to F(C^2) \to \cdots$$

Then $RF(M) = F(C^\bullet) = F(C^\bullet)$.

8.2. Composing functors. —

Proposition 8.15. — Suppose $F\colon \mathtt{A} \to \mathtt{B}$ is left exact and $G\colon \mathtt{B} \to \mathtt{C}$ is also left exact. Suppose $\mathtt{I}$ is an F-injective subcategory and $\mathtt{J}$ is a G-injective subcategory. Suppose also $F(\mathtt{I}) \subset F(\mathtt{J})$. Then $\mathtt{I}$ is $G \circ F$-injective and $R(G \circ F) = RG \circ RF$.

This statement actually replaces spectral sequences. But we will not go into this right now.

8.3. Functors of two variables. — We have two examples of functors of two variables: $\otimes$ and Hom. Deriving these deserves some detail, however we will just skip to the conclusion.

8.3.1. Tensor. — Say $\mathtt{A} = \mathtt{Mod}(R)$ and consider $F(-,-) = - \otimes_R -$. This functor is right exact in both variables. To compute $LF(M, N)$ we resolve either M or N by projective modules. This turns out to give a well defined functor $LF\colon \mathtt{D}^-(R) \times \mathtt{D}^-(R) \to \mathtt{D}^-(R)$. We write $M \overset{L}{\otimes} N$ for the derived tensor product and write $\mathrm{Tor}_i^R(M, N) = H^{-i}(M \overset{L}{\otimes} N)$.

More explicitly, say $E, F \in \mathtt{K}^-(\mathtt{A})$ are two complexes. We define the tensor product $E \otimes F$ as

$$(E \otimes F)^k = \bigoplus_{p+q=k} E^p \otimes F^q \tag{8.2}$$

$$d(x \otimes y) = dx + (-1)^k dy \tag{8.3}$$

To compute $E \overset{L}{\otimes} F$ we need to replace E or F by a quasi-isomorphic complex P where P^k is projective for all k.

In the global case, i.e. when $\mathtt{A} = \mathtt{Mod}(\mathcal{O}_X)$ or $\mathtt{Coh}(X)$ then projective objects will not be available. Nevertheless, it suffices to resolve either variable by vector bundles or flat sheaves (not to be confused by vector bundles with a flat connection).

8.3.2. Hom. — This functor is a little trickier, as it is covariant in one variable and contravariant in the other. In any case, the result is the same: if $\mathtt{A}$ has enough injectives (which is the case for $\mathtt{A} = \mathtt{Mod}(R)$ or, more generally, $\mathtt{A} = \mathtt{Mod}(\mathcal{O}_X)$) we just resolve one variable by injectives.

Exercise 8.16. — Consider the functor $F_M(-) = \mathrm{Hom}(M, -)$ for some fixed object $M \in \mathtt{A}$. Suppose there is a subcategory $\mathtt{B} \subset \mathtt{A}$ which is F_M-injective for any M. Then $\mathtt{B}$ is the subcategory of injective objects. Similarly for $G_N(-) = \mathrm{Hom}(-, N)$: the only category which works is the category of projective objects. Since $\mathtt{Mod}(\mathcal{O}_X)$ will not have enough projectives in general, we focus on the case of injectives.

For E, F complexes, we define the Hom-complex $\mathrm{Hom}^\bullet(E, F)$ by

$$\mathrm{Hom}^k(E, F) = \bigoplus_{q-p=k} \mathrm{Hom}(A^p, B^q) \tag{8.4}$$

$$d_{\mathrm{Hom}}(f) = d_F f - (-1)^k f d_E. \tag{8.5}$$

To compute $\mathrm{RHom}(E, F)$ we find I a complex of injectives quasi-isomorphic to F and compute $\mathrm{RHom}(E, F) = \mathrm{Hom}^\bullet(E, I)$.

We write

$$\mathrm{Ext}^i_{\mathtt{A}}(E, F) = H^i(\mathrm{RHom}(E, F)). \tag{8.6}$$

Proposition 8.17. — Assume $\mathtt{A}$ has enough injectives. Then

$$\mathrm{Ext}^i_{\mathtt{A}}(E, F) = \mathrm{Hom}_{\mathtt{D}(\mathtt{A})}(E, F[i]). \tag{8.7}$$

Proof. — To compute Ext, we resolve F by a complex of injectives I. Then $\mathrm{Hom}_{\mathtt{D}}(E, F[k]) = \mathrm{Hom}_{\mathtt{K}}(E, I[k])$. We notice that an element of $Z^k(\mathrm{Hom}^\bullet(E, I))$ is the same as a chain map

$E \to I[k]$. Moreover, an element f in $\mathrm{Hom}^k(E, I[k])$ is a boundary if and only if it is null homotopic. Hence,

$$\mathrm{Ext}^k(E,F) = H^k(\mathrm{Hom}^\bullet(E,I)) = \mathrm{Hom}_K(E, I[k]) = \mathrm{Hom}_D(E, I[k]) = \mathrm{Hom}_D(E, F[k]). \tag{8.8}$$

□

8.3.3. Yoneda Ext. — This realization makes the "cup-product" in Ext transparent. There is a pairing

$$\mathrm{Ext}^i(E,F) \times \mathrm{Ext}^j(F,G) \to \mathrm{Ext}^{i+j}(E,G) \tag{8.9}$$

given by composition. If we have $E \to F[i]$ and $F \to G[j]$ (which is the same as $F[i] \to G[i+j]$) we obtain $E \to G[i+j]$.

Exercise 8.18. — Let $A, B \in \mathtt{A}$. Show there is a bijection between $\mathrm{Hom}_D(A, B[1])$ and short exact sequences (a.k.a. extensions)

$$0 \to B \to E \to A \to 0$$

Moreover, using Baer sums, this becomes an isomorphism of abelian groups.

More generally, there is a bijection between $\mathrm{Hom}_D(A, B[n])$ and exact sequences

$$0 \to B \to E_1 \to \cdots \to E_n \to A \to 0. \tag{8.10}$$

This is ultimately due to the fact that given $A \to B[n]$ we may embed it in an exact triangle $\to A \to B[n] \to E \to A[1]$. Take a chain complex representing E, we may look at the long exact sequence in cohomology, which tells us that the cohomology of E is A in one degree, B in another and zero everywhere else. In other words, E can be represented by a complex

$$0 \to E_1 \to \cdots \to E_n \to 0$$

making the sequence

$$0 \to B \to E_1 \to \cdots \to E_n \to A \to 0$$

exact.

Certainly, the homological algebra we covered deserved more detail and care. However, in the interest of time, we will content ourselves with we have learned so far and move on to different things.

9. Algebraic propaganda

This section is not entirely rigorous and is completely unnecessary. We assume some knowledge of smooth manifolds and vector bundles.

References:

- Nestruev, Smooth manifolds.
- Notes on Nick Addington's website.

Let's start with an old theorem. If X is a smooth manifold, write $\mathbf{R}[X]$ with the set of *smooth* functions $f : X \to \mathbf{R}$. The field of real numbers $\mathbf{R}$ includes in $\mathbf{R}[X]$ as the subset of constant functions. Any two functions $f, g \in \mathbf{R}[X]$ can be added or multiplied together. Thus $\mathbf{R}[X]$ has the structure of an algebra over $\mathbf{R}$.

***Theorem 9.1* (cf Nestruev's book on manifolds).** — Suppose X and Y are smooth manifolds. Assume there is an isomorphism of $\mathbf{R}$-algebras $\mathbf{R}[X] \simeq \mathbf{R}[Y]$. Then X is diffeomorphic to Y.

The theorem says that functions on a manifold completely determine the manifold itself. Actually, more is true. Any smooth map $\phi\colon X \to Y$ induces a morphism (of $\mathbf{R}$-algebras) $\mathbf{R}[Y] \to \mathbf{R}[X]$ by sending $g\colon Y \to \mathbf{R}$ to $\phi^*(g) = g \circ \phi\colon X \to \mathbf{R}$. This "upper star" procedure, defines a functor from $\mathtt{Man}$, the category of smooth manifolds, to $\mathtt{Alg}_{\mathbf{R}}$, the category of *commutative* $\mathbf{R}$-algebras.

Technically, the "upper star" functor does not go from $\mathtt{Man}$ to $\mathtt{Alg}_{\mathbf{R}}$ as it sends a morphism $X \to Y$ to a morphism $\mathbf{R}[Y] \to \mathbf{R}[X]$, thus swapping the order of X and Y. To fix this, we either define *contravariant* functors or we change the codomain from $\mathtt{Alg}_{\mathbf{R}}$ to $\mathtt{Alg}_{\mathbf{R}}^{\mathrm{op}}$, the *opposite* category.

Remark 9.2. — Given a category $\mathtt{C}$, we write $\mathtt{C}(X, Y)$ or $\mathrm{Hom}_{\mathtt{C}}(X, Y)$ or $\mathrm{Mor}_{\mathtt{C}}(X, Y)$ for the set of morphisms between the objects X and Y. We define the opposite category $\mathtt{C}^{\mathrm{op}}$ to have the same objects as $\mathtt{C}$ but where $\mathtt{C}^{\mathrm{op}}(Y, X) = \mathtt{C}(X, Y)$. To avoid going insane, a good rule is to *never* write morphisms in the opposite category.

The more refined version of the theorem above is the following.

Theorem 9.3. — The functor $\mathtt{Man} \to \mathtt{Alg}_{\mathbf{R}}^{\mathrm{op}}$ sending X to $\mathbf{R}[X]$ is fully faithful.

This theorem says that differential geometry is *affine*, i.e. the space X and its algebra of functions $\mathbf{R}[X]$ are interchangeable.

Remark 9.4. — Let $F\colon \mathtt{C} \to \mathtt{D}$ be a functor and let X, Y be two objects in $\mathtt{C}$. Since morphisms are sent to morphisms, F induces a function $i\colon \mathtt{C}(X, Y) \to \mathtt{D}(F(X), F(Y))$. We say F is *full* if i is surjective, *faithful* if i is injective.

We say F is *essentially surjective* if any object $Z \in \mathtt{D}$ is isomorphic to an object $F(X)$.

There is also a theorem characterizing vector bundles in algebraic terms, but we'll get back to that later.

9.1. Going back. — The proof of the theorems above is contained in a book by Nestruev and will not be reproduced here. But let's at least catch a glimpse of the argument, at least when X is compact. Suppose $A = \mathbf{R}[X]$ is the ring of functions on the manifold X. This ring contains many interesting ideals. Indeed, let $S \subset X$ be any subset, we can consider $I_S \subset A$ to be

$$I_S = \{f : X \to \mathbf{R} \mid f(s) = 0, \forall s \in S\}$$

in other words, $f|S \equiv 0$. Clearly, if $T \subset S$ then $I_S \subset I_T$. The minimal subsets of X are points, and these correspond to maximal ideals.

Proposition 9.5. — Let $x \in X$. Then I_x, the ideal of functions vanishing at x, is maximal.

Proof. — Consider the ring homomorphism $\mathbf{R}[X] \to \mathbf{R}$ sending f to $f(x)$. It is surjective as $\mathbf{R}[X]$ contains the constant functions. Its kernel is precisely I_x. Hence $\mathbf{R} = \frac{\mathbf{R}[X]}{I_x}$, hence I_x is maximal. □

Proposition 9.6. — Let $I \subset \mathbf{R}[X]$ be a maximal ideal. Then $I = I_x$ for some $x \in X$.

Proof. — Consider S to be the set of $x \in X$ such that $f(x) = 0$ for all $f \in I$. Assume $S = \emptyset$.

Then, for all $y \in X$ we can find f_y such that $f_y(y) \neq 0$. In particular, we can find U_y a whole neighbourhood of y where $f_y|U_y$ is nowhere zero. By compactness, we may choose $f_1, \cdots, f_n$ such that $U_1 \cup \cdots \cup U_n = X$ and $f_j|U_j$ is nowhere zero. On the other hand, $f = f_1^2 + \cdots + f_n^2$ is nowhere zero and belongs to I. But then $1/f$ is well defined, contradicting maximality of I. Hence S must contain at least one point x. In particular $I \subset I_x$ and, by maximality, $I = I_x$. □

Sometimes the set of maximal ideals of a ring A is denoted MSpec A.

9.2. Vector bundles. — Let X be a manifold and let A be the ring of smooth functions. Let $E \to X$ be a smooth vector bundle on a smooth manifold. The set of sections is denoted by $\Gamma(E, X)$. Since each fibre E_x is intrinsically a vector space, $\Gamma(E)$ inherits a lot of structure: it is a module over A. The reason is intuitive: if $s, t \colon X \to E$ are sections and $f \colon X \to \mathbf{R}$ is a function then $f(x)s(x) + t(x)$ makes sense fibrewise and varies smoothly as x changes.

It's not surprising that we want to phrase this in categorical terms. Vector bundles on X form a category, $\mathtt{Vect}_X$. The objects are vector bundles $E \to X$ and the morphisms are smooth maps $E \to F$ which are linear on fibres and commute with the projection to X. Modules over A also form a category, $\mathtt{Mod}(A)$.

***Theorem 9.7* (sometimes called Serre-Swan).** — Taking global sections produces a functor

$$\Gamma \colon \mathtt{Vect}_X \to \mathtt{Mod}(A) \tag{9.1}$$

This functor is fully faithful. Its essential image consists of modules which are projective.

Recall that a module M is projective if and only if there is another module N, an integer n and an isomorphism $M \oplus N \simeq A^n$. In other words, M is projective if it is a factor of a free module.

Remark 9.8. — Given a functor $F \colon \mathtt{C} \to \mathtt{D}$, we define the *essential image* of F to be the set of objects $Y \in \mathtt{D}$ such that there exists an object $X \in \mathtt{C}$ and an isomorphism $F(X) \simeq Y$.

Let us write $\mathtt{Proj}(A)$ for the category of projective modules. Global sections provides an identification $\mathtt{Vect}(X) \simeq \mathtt{Proj}(A)$ and the diagram

$$\begin{array}{ccc} \mathtt{Vect}(X) & \xrightarrow[\sim]{\Gamma} & \mathtt{Proj}(A) \\ \downarrow & & \downarrow \\ ? & \dashrightarrow & \mathtt{Mod}(A) \end{array}$$

begs the question of whether there is some bigger category containing $\mathtt{Vect}(X)$, playing the role of general modules. In algebraic geometry, the answer is provided by $\mathtt{Coh}(X)$, the category of *coherent sheaves.*

9.3. Not everything is affine. — Consider now $X = S^2$. Since X is diffeomorphic to $\mathbf{CP}^2$, we can view X as either a smooth or complex manifold. The section above tells us that, as a smooth manifold, X and the of functions on it are interchangeable. However, the ring $\mathcal{H}(X)$ of global holomorphic functions $f : X \to \mathbf{C}$ contains no information whatsoever (well, we can deduce from it that X is non-empty and connected). Indeed, basic complex analysis tells us that $\mathcal{H}(X) = \mathbf{C}$, i.e. the only global holomorphic functions are the constant ones. This tells us that complex geometry is *not* affine. Algebraic geometry is the same. However, algebraic varieties (and a similar story holds for complex manifolds as well) can be covered by local pieces which are affine.

9.4. GAGA. — The extrinsic approach to studying varieties starts with complex projective space $\mathbf{P}^N = \mathbf{CP}^N$. This dude is a complex manifold. A *smooth projective variety* is a complex manifold X admitting a holomorphic closed embedding $X \subset \mathbf{P}^N$. It is an important fact that any such X is actually *algebraic*. In other words, there exist homogeneous polynomials $f_1, \ldots, f_r$ such that $X = \{p \in \mathbf{P}^N \mid f_1(p) = 0 = f_2(p) = \cdots = f_r(p)\}$. But more is true, any holomorphic map between such varieties is algebraic. To summarize: we can forget about the complex structure entirely and think in algebraic terms.

Here is a context where this line of thought is useful. Serre's GAGA theorem says that any holomorphic vector bundle E on X is algebraic. In particular, this means E can be trivialized on a *Zariski* open cover! This is not obvious at all from the definitions.

Remark 9.9. — The Zariski topology on $\mathbf{P}^N$ has for basis the subsets

$$D_+(f) = \{p \in \mathbf{P}^N \mid f(p) \neq 0\}$$

for f a homogeneous polynomial. The Zariski topology on $X \subset \mathbf{P}^N$ is the induced one. This is very different from the ordinary topology. Indeed, basic complex analysis shows that on $\mathbf{P}^1$ it's the same as the cofinite topology.

9.5. From bundles to sheaves. — It is important to generalize the notion of vector bundle. Before we explain why, let's fix some notation. If X is a variety (projective or otherwise) we write $\mathcal{O}_X$ for the trivial vector bundle $X \times \mathbf{C}$. More generally, if V is a vector space over $\mathbf{C}$, we write $\mathcal{O}_X \otimes_{\mathbf{C}} V$ for the trivial vector bundle $X \times V$. If $E \to X$ is a vector bundle, and $x \in X$, we write $E(x)$ for the fibre of E over x. This fibre has the structure of a complex vector space. A map of vector bundles $E \to F$ on X, induces a linear map on the fibres $E(x) \to F(x)$ for all x. Given two vector bundles E, F their tensor product will be denoted by $E \otimes_X F$ or (more often) just $E \otimes F$.

Let V be a complex vector space and let $\mathbf{P} = \mathbf{P}(V)$ be its projectivization. We can view elements of $\mathbf{P}$ as 1-dimensional subspaces $l \subset V$ (i.e. lines through the origin). There is an important vector bundle on $\mathbf{P}$, called the tautological bundle and denoted $\mathcal{O}_{\mathbf{P}}(-1)$. It can be defined explicitly as a sub-bundle of $\mathcal{O}_{\mathbf{P}} \otimes_{\mathbf{C}} V$ as follows.

$$\mathcal{O}_{\mathbf{P}}(-1) = \left\{(l, v) \in \mathbf{P} \times V \mid v \in l\right\} \subset \mathcal{O}_{\mathbf{P}} \otimes_{\mathbf{C}} V$$

Remark 9.10. — As a sidenote, the quotient of this inclusion is also a special vector bundle on $\mathbf{P}$. Indeed, the *Euler sequence* is the short exact sequence of vector bundles

$$0 \to \mathcal{O}_{\mathbf{P}}(-1) \to \mathcal{O}_{\mathbf{P}} \otimes_{\mathbf{C}} V \to T_{\mathbf{P}}(-1) \to 0$$

where $T_{\mathbf{P}}(-1) = T_{\mathbf{P}} \otimes_X \mathcal{O}_{\mathbf{P}}(-1)$ by definition and $T_{\mathbf{P}}$ is the tangent bundle of $\mathbf{P}$. By tangent bundle we mean the algebraic tangent bundle, which coincides with the holomorphic tangent bundle of X when viewed as a complex manifold.

But let's not get sidetracked. Let $f \in V^*$ be a non-zero linear form. Its kernel defines a hyperplane $W < V$ and we write $H = \mathbf{P}(W)$ for its projectivization.

There is a vector bundle map $\phi\colon \mathcal{O}(-1) \to \mathcal{O}$ defined by taking (l, v) to $(l, f(v))$. In other words, ϕ acts as f fibrewise. Let us analyze the induced linear maps. Fix $l \in \mathbf{P}^N$.

Suppose $l \notin H$. Then v is sent to $f(v)$. We know that $f(v) = 0$ iff $v \in W$. Since $v \notin W$, as $v \in l \subsetneq W$, we have $f(v) = 0$ implies $v = 0$. Hence ϕ is an isomorphism on fibres, away from H.

On the other hand, assume $l \in H$. A vector v in the fibre lives in W. Hence $f(v) = 0$ always. In other words, ϕ acts as the zero map fibres, when restricted to H.

Here comes the problem. Fibre by fibre, we could take the kernel (or cokernel) of the map $E_l \to F_l$. On $\mathbf{P} \setminus H$ it would be the zero vector space, but on H it would be $\mathbf{C}$. Globally, this kernel (or cokernel) would have to be some sort of vector bundle where the rank is zero almost everywhere and jumps to one on H. To makes sense of these spiky vector bundles, we need the notion of *sheaf.*

10. Sheaves, for real now

References:

- Kempf, Algebraic Varieties

Let X be a manifold. If U is an open subset, let $C^\infty(U)$ be the ring of smooth functions $U \to \mathbf{R}$. If $V \subset U$ we have a restriction map $C^\infty(U) \to C^\infty(V)$. This data is what is called a *presheaf.*

Definition 10.1. — More formally, there is a category $\mathtt{Op}(X)$ where objects are open subsets $U \subset X$ and $\mathtt{Op}(V, U)$ is a singleton if $V \subset U$ and empty otherwise. If $\mathtt{C}$ is any other category (for example sets, abelian groups, rings, $\mathbf{C}$-vector spaces or $\mathbf{C}$-algebras) a $\mathtt{C}$-presheaf is a functor $\mathtt{Op}(X)^{op} \to \mathtt{Set}$. More generally, the category of presheaves is the category of functors $\mathtt{Op}(X)^{op} \to \mathtt{C}$. We write $\mathtt{PSh}(X, \mathtt{C})$.

Going back to our example: C^∞ is moreover a *sheaf,* which means local data can be glued to form a global datum. Concretely, say $\{U_i\}_i$ is an open cover of some open U and that, moreover, $f_i \in C^\infty(U_i)$ are functions such that $f_i|U_{ij} = f_j|U_{ji}$ then there exists a unique function $f \in C^\infty(U)$ such that $f|U_i = f_i$. Here $U_{ij} = U_i \cap U_j$.

On the other hand, take $X = \mathbf{R}$ and write $B(U) \subset C^\infty(U)$ for the subring of smooth *bounded* functions. This G is *not* a sheaf. Indeed, take the cover $U_n = (-n, n)$ of $\mathbf{R}$. Take f_n to be $f_n(x) = x$. Then $f_i|U_{ij} = f_j|U_{ij}$ but there is no global $f \in B(\mathbf{R})$, because the identity is not bounded on $\mathbf{R}$. We write $\mathtt{Sh}(X, \mathtt{C}) \subset \mathtt{PSh}(X, \mathtt{C})$ for the subcategory of sheaves.

Digression 10.2. — As mentioned earlier, we could view $\mathtt{Sh}$ as a localization of $\mathtt{PSh}$. Indeed, declare a morphism $F \to G$ of presheaves to be a *local-isomorphism* if there exists an open cover U_i such that the induced map (of presheaves on U_i) $F|U_i \to G|U_i$ is an isomorphism for all i. Then $\mathtt{Sh}(X, \mathtt{C}) = \mathtt{PSh}(X, \mathtt{C})[\text{local-iso}^{-1}]$.

Digression 10.3. — Note that for the result above to be true one might need some assumptions on the category $\mathtt{C}$ (which are all satisfied in the cases we are considering). If instead of X one is working in an arbitrary site (for example the étale site of a variety) one should be wary of set-theoretic difficulties (which for example occur for the fpqc site).

Another natural source of sheaves are vector bundles. Let $\pi\colon E \to X$ be a smooth vector bundle. Then we define $\Gamma(U, E)$ to be the set of smooth maps $s\colon U \to E$ such that $\pi s = \mathrm{id}_U$. (these maps are typically called *sections* of the vector bundle E) This $\Gamma(-, E)$ is a sheaf on X.

10.1. Sheaves of modules. — Notice that our first example C^∞ was a sheaf where each $C^\infty(U)$ was a ring and all maps $C^\infty(U) \to C^\infty(V)$ were ring homomorphisms. On the other hand, $\Gamma(U, E)$ is not a ring. But Γ has a module structure. Indeed, given a smooth function $f \in C^\infty(U)$ and a section $s \in \Gamma(U, E)$ the multiplication $fs \in \Gamma(U, E)$ is well defined. In other words, $\Gamma(U, E)$ is a module over the ring $C^\infty(U)$. We say C^∞ is a sheaf of rings and $\Gamma(-, E)$ is a sheaf of C^∞-modules.

The mantra of sheaves of modules is that they are just like modules. We will take everything we can with modules over a ring and just do the same thing one open subset $U \subset X$ at a time.

Suppose X is a topological space and $\mathcal{O}_X$ is a sheaf of rings. For example, X a smooth manifold and $\mathcal{O}_X(U)$ is the set of smooth functions $f\colon U \to \mathbf{R}$, for $U \subset X$. Or maybe, X a complex manifold and $\mathcal{O}_X(U)$ is the set of holomorphic functions $f\colon U \to \mathbf{C}$. Since $\mathcal{O}_X$ is a sheaf of rings, we can define sheaves $\mathcal{O}_X$-modules. These form a category $\mathtt{Mod}(\mathcal{O}_X)$.

Definition 10.4. — Suppose X is a topological space and $\mathcal{O}_X$ is a sheaf of rings. Let F be a sheaf on X. We say F is a sheaf of $\mathcal{O}_X$-modules if F(U) is an $\mathcal{O}_X(U)$-module and for $V \subset U$ the restriction map $F(U) \to F(V)$ is $\mathcal{O}$-linear.

A morphism $F \to G$ of $\mathcal{O}_X$-modules consists of $\mathcal{O}_U$-linear maps $F(U) \to G(U)$ such that, for any $V \subset U$, the compositions $F(U) \to F(V) \to G(V)$ and $F(U) \to G(U) \to G(V)$ coincide.

We write $\mathtt{Mod}(\mathcal{O}_X)$ for the category of $\mathcal{O}_X$-modules and write $\mathrm{Hom}_X(F, G)$ for the set of morphisms of sheaves of $\mathcal{O}_X$-modules from F to G. Notice that $\mathrm{Hom}_X(\mathcal{O}_X, F) = F(X)$ for any sheaf F.

As the name suggests, $\mathtt{Mod}(\mathcal{O}_X)$ is an abelian category. However, one must be a little careful when defining images, cokernels and the like.

Proposition 10.5. — The category $\mathtt{Mod}(\mathcal{O}_X)$ is abelian. If $f\colon M \to N$ is a morphism, then $\ker f$ is the sheaf defined by $\ker f(U) = \ker(M(U) \to N(U))$. However, the cokernel $\operatorname{coker} f$ is defined to be the sheafification of the presheaf which assigns to U the module $\operatorname{coker}(M(U) \to N(U))$. Similarly with images.

Remark 10.6. — The problem is that coker, when defined naively might fail to be a sheaf. However, a presheaf F can always be turned into a sheaf by a process called sheafification.

Remark 10.7. — Sheafification can be concisely defined as follows. Let $\iota\colon \mathtt{Sh} \to \mathtt{PSh}$ be the inclusion of sheaves inside of presheaves. There is a functor $\mathtt{PSh} \to \mathtt{Sh}$ called *sheafification* wich takes a presheaf F and sends it to the "associated sheaf" ${}^a\mathrm{F}$. The associated sheaf is characterized by the following 'adjunction' property

$$\mathtt{Sh}({}^a\mathrm{F}, \mathrm{G}) = \mathtt{PSh}(\mathrm{F}, \iota\mathrm{G}) \tag{10.1}$$

for any presheaf F and sheaf G. Concretely, this means the following: there is a distinguished map $\mathrm{F} \to^a \mathrm{F}$ (called the *unit* of the adjunction) such that: for any map $\phi\colon \mathrm{F} \to \mathrm{G}$, where G is a sheaf, there exists a unique map ${}^a\mathrm{F} \to \mathrm{G}$ such that the composition $\mathrm{F} \to^a \mathrm{F} \to \mathrm{G}$ is ϕ.

To prove sheafification exists is a little messy. The actual construction is important, but irrelevant for our purposes.

Remark 10.8. — We can't talk about sheaves without mentioning the example which presumably started it all: the exponential sequence. Suppose X is a complex manifold and let $\mathcal{O}_\mathrm{X}$ be the sheaf of holomorphic functions. Write $\mathcal{O}_\mathrm{X}^\times \subset \mathcal{O}_\mathrm{X}$ for the subsheaf consisting of *invertible* holomorphic functions: $\mathcal{O}_\mathrm{X}^\times(\mathrm{U}) = \{f\colon \mathrm{U} \to \mathbf{C} \mid f(x) \neq 0, \forall x \in \mathrm{U}\}$. Write $\mathbf{Z}_\mathrm{X}$ for the *constant sheaf* with stalk $\mathbf{Z}$:

$$\mathbf{Z}_\mathrm{X}(\mathrm{U}) = \{f\colon \mathrm{U} \to \mathbf{Z} \mid f \text{ is locally constant }\}$$

Since constant functions are holomorphic, we have an inclusion $\mathbf{Z}_\mathrm{X} \subset \mathcal{O}_\mathrm{X}$. Given $f \in \mathcal{O}_\mathrm{X}(\mathrm{U})$, we can consider $\exp(f) \in \mathcal{O}_\mathrm{X}^\times(\mathrm{U})$. Thus we have a sequence

$$0 \to \mathbf{Z}_\mathrm{X} \to \mathcal{O}_\mathrm{X} \to \mathcal{O}_\mathrm{X}^\times \to 0 \tag{10.2}$$

Now, for any $\mathrm{U} \subset \mathrm{X}$, $\mathbf{Z}_\mathrm{X}(\mathrm{U}) \to \mathcal{O}_\mathrm{X}(\mathrm{U})$ is injective. On the other hand, $\mathcal{O}_\mathrm{X}(\mathrm{U}) \to \mathcal{O}_\mathrm{X}^\times(\mathrm{U})$ is typically *not* surjective: in general we cannot take log of an invertible function. However, when U is simply connected, any function $g\colon \mathrm{U} \to \mathbf{C}^\times$ has a logarithm, i.e. a function $f\colon \mathrm{U} \to \mathbf{C}$ such that $g = \exp(f)$. So we see that, *locally on* X, the map $\mathcal{O}_\mathrm{X} \to \mathcal{O}_\mathrm{X}^\times$ is surjective. We say that the sequence 10.2 is an exact sequence of sheaves (although it's typically not exact as a sequence of presheaves).

10.1.1. Hom and Tensor. — If $\mathrm{U} \subset \mathrm{X}$ is open, we define the structure sheaf $\mathcal{O}_\mathrm{U}$ of U as $\mathcal{O}_\mathrm{U}(\mathrm{V}) = \mathcal{O}_\mathrm{X}(\mathrm{V})$. If F is a sheaf on X we define $\mathrm{F}|\mathrm{U} \in \mathtt{Mod}(\mathcal{O}_\mathrm{U})$ as $\mathrm{F}|\mathrm{U}(\mathrm{V}) = \mathrm{F}(\mathrm{V})$. Given F, G we can also define the *inner hom* as the sheaf $\underline{\mathrm{Hom}}_\mathrm{X}(\mathrm{F}, \mathrm{G})$ defined by

$$\underline{\mathrm{Hom}}_\mathrm{X}(\mathrm{F}, \mathrm{G})(\mathrm{U}) = \mathrm{Hom}_\mathrm{U}(\mathrm{F}|\mathrm{U}, \mathrm{G}|\mathrm{U}).$$

Notice that $\underline{\mathrm{Hom}}_\mathrm{X}(\mathcal{O}_\mathrm{X}, \mathrm{F}) = \mathrm{F}$ for any sheaf F.

The category $\mathtt{Mod}(\mathcal{O}_\mathrm{X})$ also has a tensor product. Indeed, given $\mathrm{F}, \mathrm{G} \in \mathtt{Mod}(\mathcal{O}_\mathrm{X})$ we can define $(\mathrm{F} \otimes_\mathrm{X} \mathrm{G})(\mathrm{U})$ as $\mathrm{F}(\mathrm{U}) \otimes_{\mathcal{O}_\mathrm{X}(\mathrm{U})} \mathrm{G}(\mathrm{U})$. Unfortunately, this is not a sheaf in general, so it must also be sheafified (however, when U is affine we don't need to sheafifiy).

Notice that $\mathrm{F} \otimes_\mathrm{X} \mathcal{O}_\mathrm{X} = \mathrm{F}$.

The functors $\underline{\mathrm{Hom}}_\mathrm{X}$ and $\otimes_\mathrm{X}$ are *adjoints* in the sense that the following holds for any sheaves A, B, C

$$\mathrm{Hom}_\mathrm{X}(\mathrm{A} \otimes_\mathrm{X} \mathrm{B}, \mathrm{C}) = \mathrm{Hom}_\mathrm{X}(\mathrm{A}, \underline{\mathrm{Hom}}_\mathrm{X}(\mathrm{B}, \mathrm{C})). \tag{10.3}$$

10.2. Sheaves and maps. — If $f\colon X \to Y$ we can 'push sheaves forward' from X to Y. If $M \in \mathtt{Sh}(X)$ is a sheaf (say of abelian groups), we define $f_*M \in \mathtt{Sh}(Y)$ as $f_*(M)(U) = M(f^{-1}(U))$.

Consider X a topological space and F_X the sheaf of (not necessarily continuous) functions from X to **C**. In other words, $F_X(U)$ is functions $U \to \mathbf{C}$. A *space with functions* is a pair $(X, \mathcal{O}_X)$ where $\mathcal{O}_X \subset F_X$ is a subsheaf.

Remark 10.9. — If $(X, \mathcal{O}_X)$ is a space with functions, then $\mathcal{O}_X(U)$ is a reduced ring. Indeed, if $f\colon U \to \mathbf{C}$ and $f^n = 0$ then $f = 0$.

Digression 10.10. — The remark above is problematic if we want to deal with multiplicities. Indeed, if $P \subset \mathbf{A}^2$ is a parabola and $L \subset \mathbf{A}^2$ is a line, the intersection $L \cap P$ generically consists of two points. However, when L is tangent to P the naive intersection consists of a single point. We saw in the introduction that we should view this as a 'point with multiplicity two'. The algebra tells us that the correct ring that comes up in this case is $R = \mathbf{C}[x]/(x^2)$, the ring of dual numbers. However, this ring cannot be modelled as a space with functions. The problem is that topologically Spec R is a single point, but the ring of functions R is two-dimensional. To deal with these more 'exotic' spaces one needs a theory more general than spaces with functions, one needs what are called *locally ringed spaces.*

Definition 10.11. — A morphism $f\colon X \to Y$ of spaces with functions is a continuous map f such that for any $g \in \mathcal{O}_Y(U)$ the composition $g \circ f\colon f^{-1}(U) \to \mathbf{C}$ belongs to $\mathcal{O}_X(U)$. In other words, f induces a map $\mathcal{O}_Y \to f_*\mathcal{O}_X$.

If $M \in \mathtt{Mod}(\mathcal{O}_X)$, the pushforward f_*F inherits the structure of a $\mathcal{O}_Y$-module by using $\mathcal{O}_Y \to f_*\mathcal{O}_X$. Namely, we have defined a functor

$$f_*\colon \mathtt{Mod}(\mathcal{O}_X) \to \mathtt{Mod}(\mathcal{O}_Y) \tag{10.4}$$

Going in the opposite direction, we can define the *pullback*

$$f^*\colon \mathtt{Mod}(\mathcal{O}_Y) \to \mathtt{Mod}(\mathcal{O}_X). \tag{10.5}$$

Its definition is algebraically cumbersome so we will content ourselves for now with the following characterization: f^* is the left adjoint of f_*. This means that for any M, N we have

$$\mathrm{Hom}_X(f^*M, N) = \mathrm{Hom}_Y(M, f_*N) \tag{10.6}$$

(although this property uniquely determines f^*, it is admittedly a little abstract).

The functor f_* is additive and left exact. While f^* is additive and right exact. Moreover, $f^*\mathcal{O}_Y = \mathcal{O}_X$ and $f^*(M \otimes_Y N) = f^*M \otimes_X f^*N$.

11. Affine varieties

The rings we will be considering here are of finite type over **C**. Concretely, A is such a ring if it can be presented as $A \simeq \mathbf{C}[x_1, \dots, x_n]/(f_1, \dots, f_r)$. Of course, any A will admit infinitely many different presentations. We will also assume A to be *reduced*, which means it has no nilpotents.

Given A, with a presentation $A = \mathbf{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$, we write Spec A for the subset of $\mathbf{C}^n$ given by the common zero locus of $f_1, \cdots, f_r$. We write $\mathbf{A}^n$ for $\operatorname{Spec} \mathbf{C}[x_1, \cdots, x_n]$.

If $S \subset A$ is any subset, we write $V(S) \subset \operatorname{Spec} A$ for the locus $p \in \operatorname{Spec} A$ such that $f(s) = 0$ for all $f \in S$.

We define the *Zariski topology* on Spec A by declaring $Z \subset \operatorname{Spec} A$ to be Zariski-closed if $Z = V(S)$ for some $S \subset A$.

If $f \in A$, we write $D(f) = \operatorname{Spec} A \setminus V(f)$. It's a good exercise to check that the $D(f)$ form a basis for the Zariski-open subsets. Moreover, any subset $W \subset \operatorname{Spec} A$ is quasi-compact (in the sense that it's compact but not necessarily Hausdorff).

Let $X = \operatorname{Spec} A$. We can turn X into a space with functions as follows. We declare $\mathcal{O}_X(D(f)) = A_f$. Since any open subset $U \subset X$ is a union of $D(f)$, the value $\mathcal{O}_X(U)$ is uniquely determined by forcing $\mathcal{O}_X$ to be a sheaf.

Remark 11.1. — Pick a presentation $A = \mathbf{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$. Since quotienting and localizing commute, A_f can be identified with polynomials g/f^n where $g \in \mathbf{C}[x_1, \ldots, x_n]$ and $g \sim g'$ if they agree mod $(f_1, \ldots, f_r)$. This is why $\mathcal{O}_X$ is a subsheaf of the sheaf of functions F_X.

Digression 11.2. — More intrinsically, if A is a reduced **C**-algebra, we could define Spec A to be the set of maximal ideals of A. Let $p < A$ be a maximal ideal, then A/p is a field. Since A is a **C**-algebra, $\mathbf{C} \to A \to A/p$ is a field extension. Since A is of finite type over **C**, a version of the Nullstellensatz says $\mathbf{C} \to A/p$ is a finite extension. Since **C** is algebraically closed, $\mathbf{C} \to A/p$ is an isomorphism: $A/p = \mathbf{C}$. If $f \in A$, we write $f(p)$ for the value of f in A/p (and we say "f evaluated at p").

Write $X = \operatorname{Spec} A$. Using the construction above, we can view A as a subset of the set of functions $X \to \mathbf{C}$.

If $S \subset A$, we write $V(S) \subset X$ to be the set of maximal ideals containing S. These are the closed subsets of the Zariski topology on X.

Dually, we write $D(f) \subset X$ for the maximal ideals p such that $f(p) \neq 0$. These subsets form a basis for the opens of the Zariski topology. We turn X into a space with functions by declaring $\mathcal{O}_X(D(f)) = A_f$.

Definition 11.3. — We say a space with functions X is an *affine variety* if it's isomorphic (as a space with functions) to Spec A for some **C**-algebra A.

If $A \to B$ is a **C**-algebra homomorphism, we have an induced map $\operatorname{Spec} B \to \operatorname{Spec} A$. Indeed, pick presentations

$$\frac{\mathbf{C}[x_1, \ldots, x_n]}{(f_1, \ldots, f_r)} = A \longrightarrow B = \frac{\mathbf{C}[y_1, \ldots, y_m]}{(g_1, \ldots, g_s)}.$$

Such a homomorphism is determined by the image of the x_i. In other words, one needs to choose n polynomials $F_1(y), \ldots, F_n(y)$ such that $f_i(F_1(y), \ldots, F_n(y)) = 0 \bmod (g_1, \ldots, g_s)$, for all i.

Dually, the induced map $\operatorname{Spec} B \to \operatorname{Spec} A$ will send the point $(a_1, \ldots, a_m)$ to $(F_1(a), \ldots, F_n(a))$. More formally, we have just defined a contravariant functor from $\mathtt{Alg}^{\mathrm{red}}_{\mathbf{C}}$ from the category of reduced $\mathbf{C}$-algebras to the category $\mathtt{S}\mathfrak{p}$ of spaces with functions. The following is perhaps the first result one sees in a first course in algebraic geometry (in some form or other).

Theorem 11.4. — The functor we just defined is fully faithful.

11.1. Quasicoherent sheaves on affines. — Let $X = \operatorname{Spec} A$ be an affine variety. If $M \in \mathtt{Mod}(A)$ we can define $\tilde{M} \in \mathtt{Mod}(\mathcal{O}_X)$ by declaring $\tilde{M}(D(f)) = M_f$. If $F \in \mathtt{Mod}(\mathcal{O}_X)$, we say F is *quasi-coherent* if $F = \tilde{M}$ for some $M \in \mathtt{Mod}(A)$. We say F is *coherent* if $F = \tilde{M}$ with M finitely generated. We write $\mathtt{Coh}(X) \subset \mathtt{QCoh}(X) \subset \mathtt{Mod}(\mathcal{O}_X)$ for these subcategories.

Proposition 11.5. — The functor $\mathtt{Mod}(A) \to \mathtt{QCoh}(X)$ sending M to $\tilde{M}$ is fully faithful and exact. Moreover, $\widetilde{(M \otimes_A N)} = \tilde{M} \otimes_X \tilde{N}$ and $\widetilde{\underline{\mathrm{Hom}}_A(M, N)} = \underline{\mathrm{Hom}}_X(\tilde{M}, \tilde{N})$.

Suppose $f\colon X \to Y$ is a morphism of spaces with functions and $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. It can be shown that $f_*(\mathtt{QCoh}(X)) \subset \mathtt{QCoh}(Y)$ and $f^*(\mathtt{QCoh}(Y)) \subset \mathtt{QCoh}(X)$. Even better, $f^*(\mathtt{Coh}(Y)) \subset \mathtt{Coh}(Y)$.

By fully faithfulness, we know f must come from a $\mathbf{C}$-algebra map $\phi\colon B \to A$. It can also be shown that

$$f^*\tilde{M} = \widetilde{(M \otimes_B A)}.$$

Its right adjoint f_* is instead the forgetful functor, which takes an A-module N and views it as a B-module.

11.2. Locally free sheaves. — We say a sheaf $F \in \mathtt{Mod}(\mathcal{O}_X)$ is *locally free* if there is an open cover U_i and isomorphisms $F|U_i \simeq \mathcal{O}_{U_i}^{\oplus n_i}$ for some n_i. It's easy to show that the number n_i is locally constant on X.

For example, take $(X, \mathcal{O}_X)$ a smooth manifold and $E \to X$ a smooth vector bundle: this will give rise to a locally free sheaf. Indeed, vector bundles are locally trivial, so there is open cover U_i of X such that $E|U_i \simeq U_i \times \mathbf{R}^{n_i}$. But the sheaf of sections of the vector bundle $U_i \times \mathbf{R}^{n_i} \to U_i$ is nothing but $\mathcal{O}_{U_i}^{\oplus n_i}$.

The correspondence actually goes both ways. If F is locally free on X, and U_i, U_j are (connected) trivializing opens, one has isomorphisms $\mathcal{O}_{U_{ij}}^{\oplus n} \to F|U_{ij} \to \mathcal{O}_{U_{ij}}^{\oplus n}$. But an isomorphism $\mathcal{O}_{U_{ij}}^{\oplus n} \to \mathcal{O}_{U_{ij}}$ is an invertible matrix with coefficients in $\mathcal{O}_{U_{ij}}$. In other words, we have defined smooth transition functions $g_{ij}\colon U_{ij} \to GL_n(\mathbf{C})$

For this reason, locally free sheaves are sometimes just called vector bundles. We write $\mathtt{Vect}(X) \subset \mathtt{Coh}(X)$ for the subcategory of (finitely generated) locally free sheaves.

11.2.1. The affine case. — Suppose $X = \operatorname{Spec} A$ is affine and M is a coherent sheaf (i.e. it's a finitely generated A-module). Then M is locally free if there is an open cover $D(f_1), \dots, D(f_n)$ with $f_i \in A$ such that $M_{f_i} \simeq A_{f_i}^{\oplus n_i}$.

Proposition 11.6. — A finitely generated module M is locally free if and only if it is projective as an A-module.

Once again we see that projective modules are the same as vector bundles.

11.3. Digression: local systems. — Let k be a field and let k_X denote the constant sheaf on X. Recall, $k_X(U)$ is the set of locally constant functions $f\colon U \to k$. A sheaf $F \in \mathtt{Mod}(k_X)$ is a *local system* (aka a *locally constant* sheaf) if it is a locally free k_X-module. The reason we have two different notions is the following.

Suppose X is a smooth manifold, there are two different sheaves of rings we'd like to consider: $\mathbf{R}_X$ (the constant sheaf) and $\mathscr{C}^\infty_X$ (the sheaf of smooth functions). If V is a vector space, we can view it as a constant sheaf $V \otimes_{\mathbf{R}} \mathbf{R}_X$. We can also consider $V \otimes_{\mathbf{R}} \mathbf{R}_X \otimes_{\mathbf{R}_X} \mathscr{C}^\infty_X$, which is the sheaf of sections of the trivial vector bundle $V \times X \to X$.

More generally, suppose L is a local system for $\mathbf{R}_X$. So, locally, $L|U = V \otimes_{\mathbf{R}} \mathbf{R}_U$ for some vector space V. Then $E = L \otimes_{\mathbf{R}} \mathscr{C}^\infty_X$ is a vector bundle on X. But it's more, it has a preferred subspace of sections. I.e., if $U \subset X$, then $L(U) \subset E(U)$.

We say a vector bundle E is a *flat vector bundle* if there is a local system L such that $L \otimes_{\mathbf{R}_X} \mathscr{C}^\infty_X = E$.

One can check that this is the same as endowing E with a flat connection (which is the same as giving E a D-module structure).

As far as I can tell, this is unrelated to the notion of flat sheaf (or flat module).

12. Varieties

Let $(X, \mathcal{O}_X)$ be a space with functions. if $U \subset X$ is open, we define $\mathcal{O}_U = \mathcal{O}_X|U$. Obviously, $(U, \mathcal{O}_U)$ is also a space with functions. We say $(X, \mathcal{O}_X)$ is a *variety* if there exists an open cover U_i, and (reduced, finite type) algebras A_i such that $(U_i, \mathcal{O}_{U_i})$ is isomorphic to $\operatorname{Spec} A_i$, as spaces with functions.

12.1. Projective varieties. — The standard example of non-affine variety is $\mathbf{P}^N$. Indeed, we have $U_i = \{[x_0, \dots, x_N] \in \mathbf{P}^N \mid x_i \neq 0\}$ is isomorphic to $\mathbf{A}^N$.

Let f be a homogeneous polynomial. For $p \in \mathbf{P}^N$, while the value $f(p)$ is not always well defined, it is well defined whether $f(p) = 0$ or $f(p) \neq 0$. Let $S \subset \mathbf{C}[x_0, \dots, x_N]$ be any collection of *homogeneous* polynomials. We define $V_+(S) \subset \mathbf{P}^N$ to be the set of all $p \in \mathbf{P}^N$ such that $f(p) = 0$ for all $f \in S$. We define $Z \subset \mathbf{P}^N$ to be Zariski-closed if $Z = V_+(S)$ for some S.

Using all this, we can endow $\mathbf{P}^N$ with the structure of a space with functions by gluing.

Exercise 12.1. — Using this definition, show that $\mathcal{O}(\mathbf{P}^N) = \mathbf{C}$.

Finally, we define a variety X to be *quasi-projective* if it's the intersection $X = U \cap Z \subset \mathbf{P}^N$ of $U \subset \mathbf{P}^N$ open and $Z \subset \mathbf{P}^N$ closed.

12.2. Back to sheaves (of course). — Since X is a variety, we can define inside $\mathtt{Mod}(\mathcal{O}_X)$ the subcategory $\mathtt{QCoh}(X) \subset \mathtt{Mod}(\mathcal{O}_X)$ of *quasi-coherent* modules. We already defined quasi-coherent modules in the affine case, so here it's just a matter of gluing. We say $F \in \mathtt{Mod}(\mathcal{O}_X)$ is quasi-coherent if, for an affine open cover U_i of X, $F|U_i = \tilde{M}_i$ for some quasi-coherent $\tilde{M}_i$. We say F is coherent if the M_i can be taken to be finitely generated.

All three categories $\mathtt{Coh}(X) \subset \mathtt{QCoh}(X) \subset \mathtt{Mod}(\mathcal{O}_X)$ are abelian and the inclusions are exact and closed kernels, cokernels and extensions. Moreover, all categories are closed under $\otimes_X$ and $\underline{\mathrm{Hom}}_X$.

If $f: X \to Y$ is a regular map, then f^* will send $\mathtt{QCoh}(Y)$ to $\mathtt{QCoh}(X)$ and also $\mathtt{Coh}(Y)$ to $\mathtt{Coh}(Y)$. On the other hand, f_* will send $\mathtt{QCoh}(X)$ to $\mathtt{QCoh}(Y)$ but typically $\mathtt{Coh}(X)$ will not be sent to $\mathtt{Coh}(Y)$. The latter happens when f is *proper*, which means that the fibres of f are compact in an appropriate sense (note: any map between projective varieties is proper).

For example, consider $X = \mathbf{A}^1$ and $f: X \to \mathtt{pt}$, the only map to a point. This corresponds to the inclusion $\mathbf{C} \to \mathbf{C}[x]$. If M is a $\mathbf{C}[x]$-module, f_*M simply views it as a $\mathbf{C}$-vector space. For example, $\mathcal{O}_X = \mathbf{C}[x]$ is infinite-dimensional as a $\mathbf{C}$-vector space, hence does not belong to $\mathtt{Coh}(\mathtt{pt})$.

12.3. Closed immersion. — Let $f: Y \to X$ be any map. By the adjunction property, for any sheaf M on X and N on Y we have two natural maps

$$M \to f_* f^* M$$
$$f^* f_* N \to N$$

called the *unit* and *counit* of the adjunction. If $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$ and f comes from $A \to B$ we have

$$M \to M \otimes_A B$$
$$m \mapsto m \otimes 1$$

and

$$N \otimes_A B \to N$$
$$n \otimes b \mapsto nb$$

which is the action of B on N given by the B-module structure of N.

We must mention an important special case. Suppose $A \twoheadrightarrow B$ is surjective, in other words $B = A/I$ for some ideal $I < A$. Then $M \otimes_A B = M/IM$ so that the counit $M \to M \otimes_A B$ is surjective. In general, suppose $i: X \hookrightarrow Y$ is a closed subvariety, then $M \to i_* i^* M$ is always a surjective map of sheaves.

If $i: X \hookrightarrow Y$ is a closed immersion (aka a closed embedding aka the inclusion of a closed subvariety), it is common practice to denote by $\mathcal{O}_X$ the sheaf $i_* \mathcal{O}_X$ on Y. Since $\mathcal{O}_X = i^* \mathcal{O}_Y$, we have a surjection $\mathcal{O}_Y \twoheadrightarrow \mathcal{O}_X$. Its kernel is denoted by I_X and is called the *ideal sheaf* of X in Y.

Once again, in the affine case this is something we understand very well. The inclusion $i: X \hookrightarrow Y$ becomes the surjection $A \twoheadrightarrow B = A/I$. The map $\mathcal{O}_Y \twoheadrightarrow \mathcal{O}_X$ is $A \twoheadrightarrow A/I$ and I_X is really the ideal I.

12.3.1. Points. — When $x \in X$ is a point, it is common to write $k(x)$ for its structure sheaf, when viewed as a sheaf on X. This is a special case of what we just discussed: if $i: \mathtt{pt} \to X$ is the inclusion of x, then $k(x) = i_* \mathcal{O}_{\mathtt{pt}}$.

***Proposition 12.2* (Nakayama).** — Let $M \in \mathtt{Coh}(X)$. Then $M = 0$ if and only if $M \otimes k(x) = 0$ for all $x \in X$.

Proof. — One direction is obvious. Suppose $M \otimes k(x) = 0$ for all $x \in X$. First, we notice that $M = 0$ if and only if there exists an open cover U_i of X and $M|U_i = 0$ for all i. Thus we can reduce to the case $X = \operatorname{Spec} A$ is affine. It is a standard fact that a module $M = 0$ if and only if the localization $M_{\mathfrak{m}} = M \otimes_A A_{\mathfrak{m}}$ is zero for all maximal ideals $\mathfrak{m}$. By Nakayama, $M_{\mathfrak{m}} = 0$ if and only if $M_{\mathfrak{m}}/\mathfrak{m}M_{\mathfrak{m}} = 0$. Since localization and quotients commute, $M_{\mathfrak{m}} = 0$ if and only if $(M/\mathfrak{m}M)_{\mathfrak{m}} = 0$. Since $M/\mathfrak{m}M \hookrightarrow (M/\mathfrak{m}M)_{\mathfrak{m}}$ is injective, we conclude that $M = 0$ if and only if $M/\mathfrak{m}M = 0$ for all maximal ideals $\mathfrak{m}$.

Notice that, if a point x corresponds to the maximal ideal $\mathfrak{m}$, then the sheaf $k(x)$ is $A/\mathfrak{m}$. Moreover, $M/\mathfrak{m}M = M \otimes_A k(x)$. □

The *support* of $M \in \mathtt{Coh}(X)$ is the set $\operatorname{supp} M$ of $x \in X$ such that $M \otimes k(x) \neq 0$. This is a closed subset of X.

Digression 12.3. — The support can also be characterized as follows. Consider the map $\mathcal{O}_X \to \underline{\operatorname{End}}_X(M)$ taking a to the endomorphism $\cdot a$ (multiplication by a, i.e. it's a scalar matrix). Its kernel $\operatorname{Ann}(M) \subset \mathcal{O}_X$ is called the *annihilator* of M. The quotient $\mathcal{O}_X/\operatorname{Ann}(M)$ defines the *scheme-theoretic* support of M. The variety (i.e. reduced scheme) corresponding to it is the $\operatorname{supp} M$.

13. Derived functors in algebraic geometry

We have functors $(f^*, f_*, \otimes, \underline{\operatorname{Hom}})$ and we can derive them. Assume X is a variety. Recall $E \in \mathtt{D}^b(\mathtt{Mod}(\mathcal{O}_X))$ if and only if $H^i(E) = 0$ for $i \gg 0$ and $i \ll 0$. The following can also be shown to be true.

Proposition 13.1. — Let $E \in \mathtt{D}^b(\mathtt{Mod}\,\mathcal{O}_X)$. Then, $E \in \mathtt{D}^b(\mathtt{QCoh}(X))$ if and only if $H^i(E) \in \mathtt{QCoh}(X)$ for all i. Similarly, $E \in \mathtt{D}(X)$ if and only if $H^i(E) \in \mathtt{Coh}(X)$ for all i.

13.1. Pushforward. —

Proposition 13.2. — Let X be a variety. The category $\mathtt{QCoh}(X)$ has enough injectives.

This means for $f: X \to Y$ we have a well defined derived pushforward $Rf_*: \mathtt{D}^+(\mathtt{QCoh}(X)) \to \mathtt{D}^+(\mathtt{QCoh}(Y))$.

Proposition 13.3. — For any map $f: X \to Y$ between varieties and any sheaf $F \in \mathtt{QCoh}(X)$, we have $R^i f_* F = 0$ for $i > \dim X$.

Here $R^i f_* F = H^i(Rf_* F) \in \mathtt{QCoh}(Y)$. In particular, we have a well-defined functor

$$Rf_*: \mathtt{D}^b(\mathtt{QCoh}(X)) \to \mathtt{D}^b(\mathtt{QCoh}(Y)). \tag{13.1}$$

When f is moreover *proper*, we have

$$Rf_*: \mathtt{D}^b(\mathtt{Coh}(X)) \to \mathtt{D}^b(\mathtt{Coh}(Y)). \tag{13.2}$$

13.2. Global sections. — Classically, if F is a sheaf on X, one writes $\Gamma(X, F)$ for the group of global sections $F(X)$. We write $\mathtt{pt}$ for a point and $\gamma \colon X \to \mathtt{pt}$ for the only map to it. Notice that $\Gamma(X, F) = \gamma_* F$.

When $X = \operatorname{Spec} A$ is affine, recall that any quasi-coherent sheaf $F \in \mathtt{QCoh}(X)$ is identified with the A-module $F(X) = \Gamma(X, F)$.

Consider now $f \colon Y \to X$ where $X = \operatorname{Spec} A$ is affine. For $F \in \mathtt{QCoh}(Y)$, we have $f_* F \in \mathtt{QCoh}(X)$. By the identification above, $f_* F$ is identified with $\Gamma(X, f_* F) \in \mathtt{Mod}(A)$. However, $\Gamma(X, f_* F) = \Gamma(Y, F)$. Hence, $f_* = \Gamma(Y, -)$ where we keep track of the A-module structure.

More formally, let $\gamma_X \colon X \to \mathtt{pt}$ and $\gamma_Y \colon Y \to \mathtt{pt}$ be the only possible maps. Recall, $\Gamma(X, -) = \gamma_{X,*}$ and similarly for Y. Since $\gamma_Y = \gamma_X \circ f$ we have $\Gamma(Y, -) = \Gamma(X, -) \circ f_*$.

13.3. Sheaf cohomology. — We write

$$H^i(X, F) = H^i(R\gamma_* F) \in \mathtt{QCoh}(\mathtt{pt}) = \mathtt{Mod}(\mathbf{C}). \tag{13.3}$$

***Theorem 13.4* (Serre).** — Let X be a variety. Then X is affine if and only if $H^i(X, F) = 0$ for all $i > 0$ and all $F \in \mathtt{QCoh}(X)$.

This has the following consequence. Let $f \colon Y \to X$ with $X = \operatorname{Spec} A$ affine. Then $Rf_* F = H^*(Y, F)$ where we keep track of the A-module structure.

13.4. Interlude: singular cohomology. — Consider now the case of X a random topological space. Let A be a ring and let $M \in \mathtt{Mod}(A)$. We define $C_k(X, M)$ to be the free A-module spanned by continuous maps $\Delta^k \to X$ where Δ^k is the standard k-simplex. The boundary maps $\partial \colon C_k \to C_{k-1}$ turn it into a chain complex. Moreover, $H_i(C_*(X, M))$ is (by definition) the singular homology of X with coefficients in M.

We write $C^k(X, M) = \operatorname{Hom}_A(C_k(X, M), A)$ for the dual A-module. Recall that $H^i(C^*(X, M))$ is (by definition) the singular cohomology of X with coefficients in M. Recall also that, given any continuous map $f \colon Y \to X$, we can precompose to obtain a map $C^k(X, M) \to C^k(X, M)$. Moreover, this turns out to be a chain map, thus inducing pullback morphisms $H^k(X, M) \to H^k(Y, M)$.

13.4.1. Sheaf of cochains. — Write A_X for the constant sheaf of rings on X with stalk A. This means, $A_X(U) = \{f \colon U \to A \mid \text{f is locally constant}\}$. Similarly, define M_X for the constant A-module with stalk M.

On X we can define a sheaf $C^k(-, M)$ by attaching to each $U \subset X$ the module $C^k(U, M)$. Restriction maps $V \subset U$ are defined using pullbacks as above. We thus have a chain complex of sheaves

$$\cdots \to 0 \to A_X \to C^0(-, M) \to C^1(-, M) \to \cdots \tag{13.4}$$

in other words, an element of $\mathtt{Ch}^+(\mathtt{Mod}(A_X))$.

Assume now X is a topological manifold. This means that X locally looks like $\mathbf{R}^n$. Since contractible spaces have no singular cohomology, this means the sequence 13.4 is an exact sequence of sheaves. In particular, the complex

$$E = \cdots \to 0 \to C^0(-, M) \to C^1(-, M) \to \cdots$$

is quasi-isomorphic to A_X. Consider the functor γ_* of global sections. We therefore have $R\gamma_*(A_X) = R\gamma_* E$.

It turns out that the sheaves $C^k(-, M)$ are γ_*-injective (they are examples of flabby sheaves). Hence,

$$H^i(X, M) := H^i(R\gamma_* M_X) = H^i(F) \tag{13.5}$$

where F is the complex obtained by applying global sections (i.e. Γ or γ_*) to $C^*(-, M)$, in other words

$$F = \cdots 0 \to C^0(X, M) \to C^1(X, M) \to \cdots \tag{13.6}$$

In summary, for X a manifold, sheaf cohomology with coefficients in the sheaf M_X is the same as ordinary singular cohomology with coefficients in M.

13.4.2. de Rham. — Let X now be a smooth manifold. Write $\mathcal{O}_X$ for the sheaf of smooth functions on X. The tangent bundle TX is a smooth vector bundle on X. The corresponding sheaf is denoted by T_X. Dually, the cotangent bundle T^*X corresponds to the sheaf $T^*_X = \underline{\mathrm{Hom}}_X(T_X, \mathcal{O}_X)$. By taking wedges (exterior powers) we obtain the sheaves of p-forms, $\mathcal{A}^p_X$. Note $\mathcal{A}^0_X = \mathcal{O}_X$. We have a chain complex of vector bundles (the *de Rham complex*)

$$0 \to \mathbf{R}_X \to \mathcal{A}^0_X \to \mathcal{A}^1_X \to \cdots \tag{13.7}$$

where $\mathbf{R}_X$ is the constant sheaf on X with stalk $\mathbf{R}$. The Poincaré lemma says that this complex is exact (because locally ever closed form is exact). In other words, the complex

$$dR_X = \mathcal{A}^0_X \to \mathcal{A}^1_X \to \cdots \tag{13.8}$$

is quasi-isomorphic to the constant sheaf $\mathbf{R}_X$. Moreover, it can be shown that the $\mathcal{A}^p_X$ are γ_*-acyclic (this is due to the existence of partitions of unity). Consider the complex of $\mathbf{R}$-vector spaces given by global forms

$$\mathcal{A}^\bullet(X) = \cdots \to 0 \to \mathcal{A}^0(X) \to \mathcal{A}^1(X) \to \mathcal{A}^2(X) \to \cdots \tag{13.9}$$

We call its cohomology $H^i(\mathcal{A}^\bullet(X)) = H^i_{dR}(X)$ the *de Rham cohomology* of X. It follows by acyclicity of $\mathcal{A}^\bullet$ that $H^i(X, \mathbf{R}_X) = H^i_{dR}(X)$. But we saw earlier that $H^i(X, \mathbf{R}_X)$ is also isomorphic to the i-th singular cohomology of X with coefficients in $\mathbf{R}$.

13.5. Pullback. — To derive $\otimes$ and f^* we need to resolve by appropriately *flat* objects. A module $M \in \mathtt{Mod}(\mathcal{O}_X)$ is *flat* if for any short exact sequence $0 \to N' \to N \to N'' \to 0$ the sequence $0 \to M \otimes N' \to M \otimes N \to M \otimes N'' \to 0$ is exact. Notice that locally free sheaves are flat.

A map $f: X \to Y$ of spaces with functions is flat f^* is exact (equivalently, $f_* \mathcal{O}_X$ is flat as an $\mathcal{O}_Y$-module).

Proposition 13.5. — The category $\mathtt{Mod}(\mathcal{O}_X)$ has enough flat sheaves.

In particular, we have derived versions of $\overset{L}{\otimes}_X$ and $Lf^*: \mathtt{D}^-(\mathtt{Mod}\,\mathcal{O}_Y) \to \mathtt{D}^-(\mathtt{Mod}\,\mathcal{O}_X)$ for any $f: Y \to X$.

13.6. Coherent sheaves (again, really? totally unexpected). — For smooth varieties, all our functors never leave the bounded world.

Proposition 13.6. — Let X be a *smooth* quasi-projective variety. Any coherent sheaf F admits a resolution

$$0 \to E_n \to E_{n_1} \to \cdots \to E_0 \to F \to 0$$

with E_i a vector bundle (i.e. locally free).

Corollary 13.7. — For X quasi-projective, the category $\mathtt{Vect}(X)$ is projective for $\otimes_X$ and f^*, for any $f\colon Y \to X$. This means that both these functors are left derivable. Moreover, since we can pick our resolution to be *bounded* we have the following well defined functors.

$$\overset{L}{\otimes}\colon \mathtt{D}^b(\mathtt{Coh}X) \times \mathtt{D}^b(\mathtt{Coh}X) \to \mathtt{D}^b(\mathtt{Coh}X)$$
$$(E, F) \mapsto E \overset{L}{\otimes} F$$

$$f^*\colon \mathtt{D}^b(\mathtt{Coh}X) \times \mathtt{D}^b(\mathtt{Coh}X) \to \mathtt{D}^b(\mathtt{Coh}Y)$$
$$E \mapsto Lf^*(E)$$

13.7. Flat and Affine. — Flat maps are nice because f^* needn't be derived. In the other direction, if f is *affine* then f_* needn't be derived (a map is affine essentially if the fibres are affine varieties). Any map between affine varieties is affine. A closed immersion $Y \hookrightarrow X$ is affine (indeed, locally on X the map looks like Spec of $A \twoheadrightarrow A/I$).

13.8. Everything is derived. — From now on, we will pretty much only deal with X a smooth projective variety. We will write $\mathtt{D}(X)$ for $\mathtt{D}^b(\mathtt{Coh}(X))$. Moreover, all functors will be *implicitly* derived. For example, f_* will denote Rf_*. When we want to refer to the underived functor, we will simply take cohomology: $H^0(f_*F)$. The only tricky part of this convention is that Hom will denote Homs in the derived category (which is a vector space), while $\mathrm{Hom}^\bullet$ will denote $\mathrm{RHom}^\bullet$, which is a chain complex of vector spaces. With this convention, $H^0(\mathrm{Hom}^\bullet(E, F)) = \mathrm{Hom}(E, F)$. On the other hand, for sheaf hom we simply write $\underline{\mathrm{Hom}}$ in place of $R\underline{\mathrm{Hom}}^\bullet$ since it creates no confusion.

13.8.1. Tor. — For example, if $M \in \mathtt{Coh}(X)$ and $x \in X$ the symbol $M \otimes k(x)$ will denote the derived tensor product. To denote the ordinary one, we write $\mathrm{Tor}_0(M, k(x))$.

Proposition 13.8. — Suppose $E \in \mathtt{D}(X)$. Then $E = 0$ if and only if $E \otimes k(x) = 0$ for all $x \in X$.

Proof. — Suppose $E \neq 0$. Since E is bounded on the right there is a maximum number q such that $H^q(E) \neq 0$. We have a triangle

$$\tau^{\leq q-1}E \to E \to H^q(E)[-q] \xrightarrow{+}$$

Remark 13.9. — If $F \in \mathtt{D}^{\leq k}(X)$, $N \in \mathtt{Coh}(X)$ then $F \otimes N \in \mathtt{D}^{\leq k}(X)$. This follows from how we explicitly derive functors by taking resolutions.

By tensoring with $k(x)$ we get a triangle

$$\tau^{\leq q-1}E \otimes k(x) \to E \otimes k(x) \to H^q(E)[-q] \otimes k(x) \xrightarrow{+}$$

Since we are assuming $E \otimes k(x) = 0$, we deduce that $\tau^{\leq q-1}E \otimes k(x) = H^q(E)[-q] \otimes k(x)[-1] = H^q(E) \otimes k(x)[-q-1]$. But $H^{q+1}(\tau^{\leq q-1}E \otimes k(x)) = 0$ so $H^0(H^q(E) \otimes k(x)) = \mathrm{Tor}_0(H^q(E), k(x)) = 0$. Hence, by Nakayama, $H^q(E) = 0$. Which is absurd. □

Definition 13.10. — We call $\operatorname{supp} E$ the set x such that $E \otimes k(x) \neq 0$. This is the same as the union of $\operatorname{supp} H^i(E)$ for all i. It's a closed subset of X.

$E = 0$ if and only if $\operatorname{supp} E = 0$.

13.9. Ext. — More often than not, we will be dealing with X smooth and projective. In this case, $\mathrm{Hom}_X(E, F)$ is a finite-dimensional vector space. Moreover, if $E, F \in \mathtt{Coh}(X)$ are sheaves, then $\mathrm{Ext}^i_X(E, F) = 0$ for $n < i < 0$. We will also see later Serre duality, which says there is an isomorphism

$$\mathrm{Ext}^i(E, F)^\vee = \mathrm{Ext}^{n-i}(F, E \otimes \omega_X). \tag{13.10}$$

Recall that $\mathrm{Ext}^i(E, F) = \mathrm{Hom}(E, F[i]) = \mathrm{Hom}(E[-i], F)$.

Proposition 13.11. — Let C be a smooth and projective curve. Let $E \in \mathtt{D}(X)$. Then $E \cong \bigoplus_i H^i(E)[-i]$, non-canonically.

Proof. — Pick a representative of E in $\mathtt{Ch}^b(\mathtt{Coh}(X))$ and induct on its length. For example, if $E = E^{-1} \to E^0$ then we have an exact triangle $H^{-1}(E)[1] \to E \to H^0(E) \to H^{-1}(E)[2]$. But since C is a curve, $\mathrm{Ext}^2(H^0(E), H^{-1}(E)) = 0$, thus the sequence above splits and $E = H^{-1}(E)[1] \oplus H^0(E)$. □

13.10. Duals. — Let V be a vector space over $\mathbf{C}$. The dual $V^\vee$ is defined as $V^\vee = \underline{\mathrm{Hom}}_{\mathbf{C}}(V, \mathbf{C})$. When V is finite dimensional, we have $(V^\vee)^\vee = V$. If V and W are two vector spaces, with V finite dimensional, then $\underline{\mathrm{Hom}}_{\mathbf{C}}(V, W) = V^\vee \otimes_{\mathbf{C}} W$.

These identities can be generalized to sheaves. If $E, F \in \mathtt{D}(X)$, recall we have $\underline{\mathrm{Hom}}_X(E, F) \in \mathtt{D}(X)$. When E is a complex of vector bundles, we don't need to resolve E to compute $\underline{\mathrm{Hom}}$ (unlike the case of $\mathrm{Hom}^\bullet$). For $E \in \mathtt{D}(X)$, we define the (derived) dual to be $E^\vee = \underline{\mathrm{Hom}}_X(E, \mathcal{O}_X)$. We have $E^{\vee\vee} = E$

Remark 13.12. — Notice that for $E \in \mathtt{D}(\mathtt{QCoh}(X))$, the double dual needn't be isomorphic to E. Here it's crucial that $H^i(E) \in \mathtt{Coh}(X)$.

We have

$$\underline{\mathrm{Hom}}_X(E, F) = E^\vee \otimes_X F \tag{13.11}$$

$$(E \otimes F)^\vee = F^\vee \otimes E^\vee \tag{13.12}$$

$$\underline{\mathrm{Hom}}(E, F)^\vee = \underline{\mathrm{Hom}}(F, E). \tag{13.13}$$

We also have a natural isomorphism

$$\underline{\mathrm{Hom}}_X(\mathcal{O}_X, E) = E. \tag{13.14}$$

We have an evaluation map

$$\underline{\mathrm{Hom}}_X(E, F) \otimes E \to F \tag{13.15}$$

As a special case, we have the *trace* (or contraction) map

$$E^\vee \otimes E \to \mathcal{O}_X. \tag{13.16}$$

13.11. Adjunction. — Tensor and Hom are adjoints

$$\mathrm{Hom}_X(E \otimes_X F, G) \simeq \mathrm{Hom}_X(E, \underline{\mathrm{Hom}}_X(F, G)). \tag{13.17}$$

which follows (by taking global sections) from the sheaf version

$$\underline{\mathrm{Hom}}_X(E \otimes_X F, G) \simeq \underline{\mathrm{Hom}}_X(E, \underline{\mathrm{Hom}}_X(F, G)). \tag{13.18}$$

13.12. Pullback. — Let $f: X \to Y$ be a map between smooth varieties. When E is a complex of vector bundles, to compute f^*E we don't need to resolve (vector bundles are f^*-projective). Moreover, we always have

$$f^*\mathcal{O}_Y = \mathcal{O}_X \tag{13.19}$$

$$f^*(E \otimes_Y F) = f^*E \otimes_X f^*F \tag{13.20}$$

$$f^*\underline{\mathrm{Hom}}_Y(E, F) = \underline{\mathrm{Hom}}_X(f^*E, f^*F) \tag{13.21}$$

$$f^*(E^\vee) = (f^*E)^\vee \tag{13.22}$$

Let's go back to the trace map.

Proposition 13.13. — Trace $E^\vee \otimes E \to \mathcal{O}_X$ is an isomorphism if and only if E is (quasi-isomorphic to) a line bundle.

Proof. — If E is a line bundle, then this is obvious.

Conversely, suppose $U \subset X$ be an open subset. Let F be the restriction of E to U. Since $E^\vee \otimes E \to \mathcal{O}_X$ is an isomorphism in $D(X)$, we have $F^\vee \otimes F \to \mathcal{O}_U$ is an isomorphism in $D(U)$. Assume $U = \mathrm{Spec}\, A$ is affine. Resolve F by projective modules. Actually, by refining U if necessary (or passing to local rings) we can write

$$F = \cdots \to A^{n_i} \to A^{n_{i+1}} \to \cdots$$

The complex $F^\vee$ is given explicitly by dualizing that complex of free modules, in other words $(F^\vee)^k = A^{n_{-k}}$. Similarly, an explicit model for $(F^\vee \otimes F)$ has

$$(F^\vee \otimes F)^k = \bigoplus_{p+q=k} A^{n_{-p}} \otimes A^{n_q}$$

And since the whole complex must be homotopy equivalent to A we see that there is i such that $A^{n_i} = A$ and $A^{n_j} = 0$ for $j \neq i$.

In other words, we have learned that $E|U = \mathcal{O}_U[k]$ for some k. Globally (if X is connected), $E = L[k]$ for some line bundle L. □

13.13. Pushforward. — Pushforward is right adjoint to pullback, even in the derived sense

$$\mathrm{Hom}_X(f^*E, F) = \mathrm{Hom}_Y(E, f_*F) \tag{13.23}$$

which actually follows from a sheaf version of adjunction

$$f_*\underline{\mathrm{Hom}}_X(f^*E, F) = \underline{\mathrm{Hom}}_Y(E, f_*F) \tag{13.24}$$

We also have the so-called *projection formula*

$$f_*(E \otimes_X f^*F) = F \otimes_Y f_*E. \tag{13.25}$$

Remark 13.14. — Suppose $A \to B$ is a ring homomorphism. Let M be a B-module and N an A-module. The projection formula says: consider $M \otimes_B (B \otimes_A N)$ as an A-module, then this is the same as $N \otimes_A M$, where M is viewed as an A-module. Indeed, this follows by 'associativity of tensor': $M \otimes_B (B \otimes_A N) = M \otimes_A N$.

Remark 13.15. — If you look up the projection formula in Hartshorne, you'll see he assumes F to be a vector bundle. This is because in his formula the functors are not derived. When F is a vector bundle, Hartshorne's formula follows by taking H^0 everywhere.

Let now $f = \gamma\colon X \to \mathtt{pt}$, (or more generally any map to an affine). Then $\gamma_* = \Gamma(X, -) = H^\bullet(X, -)$ is the same as (derived) global sections (aka sheaf cohomology).

$$\gamma_*(F) = \Gamma(X, F) = H^\bullet(X, F) = \mathrm{Hom}^\bullet(\mathcal{O}_X, F) \tag{13.26}$$

13.14. Serre duality. — Let $L \in \mathtt{D}(X)$. We have a functor $\mu_L\colon \mathtt{D}(X) \to \mathtt{D}(X)$ which takes E and sends it to $\mu_L(E) = L \otimes E$.

Let $L^\vee = \underline{\mathrm{Hom}}_X(L, \mathcal{O}_X)$ be the dual. Then $\mu_{L^\vee}\mu_L(E) = L^\vee \otimes L \otimes E = E$. In other words, μ_L is an equivalence with inverse $\mu_{L^\vee}$.

Let $\omega_X = \bigwedge^n \Omega_X$, where $n = \dim X$. Here Ω_X can be seen either as the holomorphic cotangent bundle of X or the sheaf of Kähler differentials on X. The sheaf ω_X is called the *canonical bundle*. Its class in Pic(X) (or Chow) is often denoted K_X. We define

$$S_X\colon \mathtt{D}(X) \to \mathtt{D}(X) \tag{13.27}$$

$$E \mapsto E \otimes \omega_X[n] \tag{13.28}$$

When X is proper, this functor has a very special property (namely, it satisfies Serre duality).

Theorem 13.16. — Let X be a smooth and proper. For any $E, F \in \mathtt{D}(X)$ we have functorial isomorphisms (of finite dimensional vector spaces)

$$\mathrm{Hom}_X(E, F)^\vee \simeq \mathrm{Hom}_X(F, S_X(F)). \tag{13.29}$$

13.14.1. Classical Serre duality. — How does this compare to ordinary Serre duality? Recall that $\mathrm{Hom}(\mathcal{O}_X, F[i]) = H^i(X, F)$. By Serre duality, $\mathrm{Hom}(\mathcal{O}, F[i])^\vee = \mathrm{Hom}(F[i], \mathcal{O} \otimes \omega[n]) = \mathrm{Hom}(F, \omega[n-i])$. Let $\gamma\colon X \to \mathtt{pt}$. Recall that

$$\begin{aligned}\mathrm{Hom}(F, \omega[n-i]) &= H^0(\gamma_*\underline{\mathrm{Hom}}(F, \omega[n-i])) \\ &= H^0(\gamma_*(F^\vee \otimes \omega[n-i])) \\ &= H^{n-i}(X, F^\vee \otimes \omega).\end{aligned}$$

In other words,

$$\mathrm{H}^i(X,F)^\vee \simeq \mathrm{H}^{n-i}(X, F^\vee \otimes \omega_X). \tag{13.30}$$

13.14.2. Calabi-Yau. — We say a variety is ***Calabi-Yau*** (in a weak sense) if $\omega_X \cong \mathcal{O}_X$ is trivial. This is the algebraic analogue for a manifold being orientable (on a smooth manifold, the orientation sheaf plays the role of the canonical bundle). For Calabi-Yaus, Serre duality takes a particularly nice form

$$\mathrm{Ext}^i(E,F)^\vee \simeq \mathrm{Ext}^{n-i}(F,E). \tag{13.31}$$

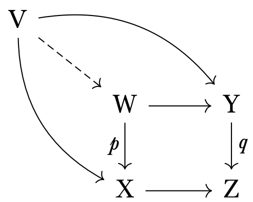
13.15. Base change. — Consider a commutative square

$$\begin{array}{ccc} W & \longrightarrow & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \longrightarrow & Z \end{array}$$

we say W is a ***fibre product*** essentially if

$$W = X \times_Z Y = \{(x,y) \in X \times Y \mid p(x) = q(y)\}$$

More formally, we can characterize fibre products with a universal property. For any variety V and maps $V \to X$, $V \to Y$ such that the compositions $V \to X \to Z$, $V \to Y \to Z$ agree there exists a unique map $V \to W$ making the diagram below commute.



Remark 13.17. — If everything in sight is affine, we can view the first square as a map of rings: $X = \operatorname{Spec} A, Y = \operatorname{Spec} B, Z = \operatorname{Spec} C, W = \operatorname{Spec} D$.

$$\begin{array}{ccc} D & \longleftarrow & B \\ \uparrow & & \uparrow \\ A & \longleftarrow & C \end{array}$$

and one checks $D = A \otimes_C B$.

Digression 13.18. — Strictly speaking, $\operatorname{Spec}(A \otimes_C B)$ is the fibre product on the category of schemes, not of varieties. To obtain a variety one would need $(A \otimes_C B)_{\mathrm{red}}$, the tensor product modulo all its nilpotent elements. We shall ignore this point.

Now, suppose M is a B-module. We can take $M \otimes_B D$ (pullback) and view it as an A-module (pushforward). On the other hand, we can view M as a C-module (pushforward) and then take $M \otimes_C A$. However,

$$M \otimes_B D = M \otimes_B (B \otimes_C A) = M \otimes_C A$$

hence the two agree.

***Proposition 13.19* (base change).** — Suppose

$$\begin{array}{ccc} W & \xrightarrow{g} & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{f} & Z \end{array}$$

is a fibre square. Suppose f is flat (and therefore g is flat). Then $p_* g^* = f^* q_*$.

13.16. Kunneth. — Consider a product $X \times Y$ with projections $p\colon X \times Y \to X$, $q\colon X \times Y$. We have a fibre square

$$\begin{array}{ccccc} & & X \times Y & & \\ & \swarrow^{p} & & \searrow^{q} & \\ X & & & & Y \\ & \searrow_{\gamma_X} & & \swarrow_{\gamma_Y} & \\ & & \mathtt{pt} & & \end{array}$$

Let $F \in \mathtt{D}(Y)$. By base change, $p_* q^* F = \gamma_X^* \gamma_{Y,*}$ thus

$$p_* q^* F = R\Gamma(Y, F) \otimes \mathcal{O}_X.$$

More generally, if $E \in \mathtt{D}(X)$, $F \in \mathtt{D}(Y)$ then

$$R\Gamma(X \times Y, p^* E \otimes q^* E) = R\Gamma(X, E) \otimes_{\mathbf{C}} R\Gamma(Y, F)$$

This follows from the projection formula.

$$\begin{aligned} R\Gamma(X \times Y, p^* E \otimes q^* E) &= \gamma_{X,*} p_* (p^* E \otimes q^* F) \\ &= \gamma_{X,*}(E \otimes_X p_* q^* F) \\ &= \gamma_{X,*}(E \otimes_X R\Gamma(Y, F) \otimes_{\mathbf{C}} \mathcal{O}_X) \\ &= \gamma_{X,*}(E \otimes_{\mathbf{C}} R\Gamma(Y, F)) \\ &= R\Gamma(X, E) \otimes_{\mathbf{C}} R\Gamma(Y, F) \end{aligned}$$

13.17. Support again. —

Proposition 13.20. — Let $E \in \mathtt{D}(X)$. Suppose $\operatorname{supp} E$ is the disjoint union of two closed subsets $Z_1 \amalg Z_2$. Then $E = E_1 \oplus E_2$ where $\operatorname{supp} E_i \subset Z_i$.

Proof. — Consider the open cover of X given by $U_i = X \setminus Z_i$. Let $j_i\colon U_i \to X$ be the inclusion and let $j\colon U_1 \cap U_2 \to X$ also denote the inclusion. Although we haven't proved this, it is a general fact that (in $\mathtt{D}(\mathtt{QCoh}(X))$) we have a *Mayer-Vietoris* triangle

$$E \to j_{1,*} j_1^* E \oplus j_{2*} j_2^* E \to j_* j^* E \xrightarrow{+}$$

Since $\operatorname{supp} E \subset Z_1 \cap Z_2$ we have $j^* E = 0$, therefore the claim follows. □

This leads to a mildly interesting result.

Definition 13.21. — Let $\mathtt{T}$ be a triangulated category and let $\mathtt{A}, \mathtt{B} \subset \mathtt{T}$ be two (non-zero) triangulated subcategories. We say $\mathtt{A}, \mathtt{B}$ are a completely orthogonal decomposition of $\mathtt{T}$ if

$\mathrm{Hom}(\mathtt{B},\mathtt{A}) = 0$
$\mathrm{Hom}(\mathtt{A},\mathtt{B}) = 0$
For any $E \in \mathtt{T}$ there is an exact triangle $A \to E \to B \to$ with $A \in \mathtt{A}$, $B \in \mathtt{B}$.

Since $\mathrm{Hom}(\mathtt{B},\mathtt{A}) = 0$ it follows that $\mathrm{Hom}(\mathtt{B},\mathtt{A}[1]) = 0$. Hence the triangle above splits and E is actually always a direct sum $E = A' \oplus B'$ with $A' \in \mathtt{A}$, $B' \in \mathtt{B}$.

Proposition 13.22. — $\mathtt{D}(X)$ has admits a completely orthogonal decomposition if and only if X is disconnected.

What will turn out to be more interesting are *semi*-orthogonal decompositions (which are defined in the same way but with the first axiom omitted).

Proof. — If $X = Z_1 \amalg Z_2$ is disconnected, then the Mayer-Vietoris triangle does the trick. Suppose instead we have an orthogonal decomposition. Let $\mathcal{O}_X = A \oplus B$ be the decomposition of the structure sheaf. Since $H^i(A) \oplus H^i(B) = H^i(\mathcal{O}_X)$, we see $A, B \in \mathtt{Coh}(X)$. Since we have a surjection $A \oplus B \to A$ (i.e. projecting onto a factor), $A = \mathcal{O}_{Z_1}, B = \mathcal{O}_{Z_2}$ must be structure sheaves of two closed subschemes. This implies X is disconnected. Indeed, assume there were $x \in Z_1 \cap Z_2$. Then $\mathrm{Hom}(\mathcal{O}_{Z_1}, k(x)) = \mathbf{C} = \mathrm{Hom}(\mathcal{O}_{Z_2}, k(x))$. But $\mathbf{C} = \mathrm{Hom}(\mathcal{O}_X, k(x)) = \mathrm{Hom}(\mathcal{O}_{Z_1}, k(x)) \oplus \mathrm{Hom}(\mathcal{O}_{Z_2}, k(x)) = \mathbf{C}^2$. □

13.18. Tensoring with vector spaces. — One thing we probably used implicitly is tensoring with vector spaces. If $E \in \mathtt{Mod}(\mathcal{O}_X)$ and $V \in \mathtt{Mod}(\mathbf{C})$ it makes to consider $V \otimes_{\mathbf{C}} E$. If we pick a basis $V \simeq \mathbf{C}^n$ then $V \otimes_{\mathbf{C}} E = E^{\oplus n}$.

Digression 13.23. — More formally, this is what's happening. A vector space V can be turned into a sheaf V_X by declaring $V_X(U)$ to be the set of locally constant functions $f\colon U \to V$. This is obviously a $\mathbf{C}_X$-module. On the other hand, $\mathbf{C}_X$ is obviously a subsheaf of rings of $\mathcal{O}_X$ (constant functions are regular).

Thus, if E is an $\mathcal{O}_X$-module, $V \otimes_{\mathbf{C}} E$ means $V_X \otimes_{\mathbf{C}_X} E$ where we view E as a $\mathbf{C}_X$-module.

This tensoring procedure obviously makes sense for complexes. Since it is exact, its derived version coincides with the underived one.

13.19. Grothendieck-Verdier duality. — Given $f\colon X \to Y$ we've seen f_* and f^*. Turns out there is a third functor $f^!$ ("f *upper shriek*"). We will only deal with $f^!$ when f is proper (and X and Y are smooth). We have $f^!\colon \mathtt{D}(Y) \to \mathtt{D}(X)$

$$f^!(E) = f^*(E) \otimes \omega_Y \otimes f^*\omega_X^{-1}[\dim Y - \dim X].$$

We have that $f^!$ satisfies the following "sheafified adjunction"

$$\underline{\mathrm{Hom}}_Y(f_*E, F) = f_*\underline{\mathrm{Hom}}_X(E, f^!F)$$

Here's another list of things which are true

$$\mathrm{Hom}_Y(f_*E, F) = \mathrm{Hom}_X(E, f^!F) \tag{13.32}$$

$$f^! = D_X \circ f^* \circ D_Y^{-1} \tag{13.33}$$

$$f^! = S_X \circ f^* \circ S_Y^{-1} \tag{13.34}$$

$$f_* \circ D_X = D_Y \circ f_* \tag{13.35}$$

where $D_X(E) = \underline{\mathrm{Hom}}_X(E, \omega_X[\dim X])$ is the *dualizing functor* and where $S_X(E) = E \otimes \omega_X[\dim X]$ is the *Serre functor.*

13.19.1. Serre duality follows from Grothendieck-Verdier duality. — Let $\gamma\colon X \to \mathtt{pt}$. Then $\omega_X[\dim X] = \gamma^!\mathbf{C}$ is called the *dualizing complex.*

Digression 13.24. — If X is proper, but not necessarily smooth, $\gamma^!$ still exists and is still an adjoint of f_*. The complex $\gamma^!\mathbf{C}$ still makes sense, but it might be a genuine complex. One can prove that X is Gorenstein if and only if $\gamma^!\mathbf{C}$ is (the shift of) a line bundle. Similarly, X is Cohen-Macaulay if and only if $\gamma^!\mathbf{C}$ is (the shift of) a sheaf.

Consider $\gamma\colon X \to \mathtt{pt}$. Then we want to compare Hom(E, F) with $\mathrm{Hom}(F, E \otimes \omega_X[\dim X])$. We know

$$\begin{aligned}
\mathrm{Hom}(E, F)^\vee &= (\gamma_*\underline{\mathrm{Hom}}(E, F))^\vee \\
&= (\gamma_*E^\vee \otimes F)^\vee \\
&= D_{\mathtt{pt}} \circ \gamma_*(E^\vee \otimes F) \\
&= \gamma_* \circ D_X(E^\vee \otimes F) \\
&= \gamma_*\underline{\mathrm{Hom}}(E^\vee \otimes F, \omega_X[\dim X]) \\
&= \mathrm{Hom}(E^\vee \otimes F, \omega_X[\dim X]) \\
&= \mathrm{Hom}(F, \underline{\mathrm{Hom}}(E^\vee, \omega_X[\dim X])) \\
&= \mathrm{Hom}(F, E \otimes \omega_X[\dim X]).
\end{aligned}$$

Digression 13.25. — When f is ètale, $f^!$ is actually just the same as f^*. To define f for an arbitrary morphism, we use the fact that any map between quasi-projective varieties factors as $f = pj$ with p proper and j an open immersion (for general schemes one needs to use Nagata compactification). Making sure everything is well defined is hard (and explained in Hartshorne's book Residues and Duality).

Digression 13.26. — In topology we also have a sixth functor $f_!$ "f *lower shriek*". The gang $(f^*, f_*, f^!, f_!, \otimes, \underline{\mathrm{Hom}})$ is called the *formalism of six functors.*

When f is proper, $f_! = f_*$. Sadly, algebraic geometry lacks this functor when f is non proper.

Actually, this is not quite correct (and it's actually part of a bigger story called *Tate geometry* which is getting attention recently). Let's go back to everything underived. If is proper, we know $f_*\colon \mathtt{Coh}(X) \to \mathtt{Coh}(Y)$. In this case, we would want $f_! = f_*$. In general, however, $f_*\colon \mathtt{Coh}(X) \to \mathtt{QCoh}(Y)$.

In the appendix to Residues and Duality, Deligne actually constructs a functor $f_!\colon \mathtt{Coh}(X) \to \mathrm{Pro}\mathtt{Coh}(Y)$, where the latter is the pro-completion of the category of coherent sheaves. This essentially means formally adding all filtered limits to $\mathtt{Coh}(Y)$. As far as I can tell, this category has received very little attention.

To understand this slightly better, we should mention the following fact. Dual to Pro, there is the ind-completion $\mathrm{Ind}\mathtt{Coh}(Y)$, which is the category obtained by formally adding all filtered colimits. It turns out (it's actually not hard to show) that $\mathrm{Ind}\mathtt{Coh}(Y) = \mathtt{QCoh}(Y)$. So, in a certain sense, the correct codomain for f_* is $\mathrm{Ind}\mathtt{Coh}(Y)$. But the latter just happens to be the very reasonable category $\mathtt{QCoh}(Y)$.

13.19.2. A special case. —

Corollary 13.27. — Suppose $i\colon Y \to X$ is the embedding of a smooth subvariety of codimension c. Then

$$(i_*\mathcal{O}_Y)^\vee = i_*\omega_Y \otimes \omega_X^\vee[-c]$$

Proof. — By the Yoneda lemma, it suffices to prove that for any $G \in \mathtt{D}(X)$ we have functorial isomorphisms

$$\mathrm{Hom}(G, (i_*\mathcal{O}_Y)^\vee) = \mathrm{Hom}(G, i_*\omega_Y \otimes \omega_X^\vee[-c])$$

So,

$$\begin{aligned}
\mathrm{Hom}(G, (i_*\mathcal{O}_Y)^\vee) &= \mathrm{Hom}(G, \underline{\mathrm{Hom}}(i_*\mathcal{O}_Y, \mathcal{O}_X)) \\
&= \mathrm{Hom}(G \otimes i_*\mathcal{O}_Y, \mathcal{O}_X) \\
&= \mathrm{Hom}(i_*(i^*G \otimes \mathcal{O}_Y), \mathcal{O}_X) \\
&= \mathrm{Hom}(i_*i^*G, \mathcal{O}_X) \\
&= \mathrm{Hom}(i^*G, i^!\mathcal{O}_X) \\
&= \mathrm{Hom}(G, i_*i^*!\mathcal{O}_X)
\end{aligned}$$

and

$$\begin{aligned}
i_*i^!\mathcal{O}_X &= i_*((i^*\omega_X^\vee[-\dim X]) \otimes \omega_Y[\dim Y]) \\
&= \omega_X^\vee \otimes i_*\omega_Y[-c].
\end{aligned}$$

□

Why is this interesting? Well, consider the case $Y = \mathtt{pt} \subset X$ is a point. Then $i_*Y = k(x)$ for some x. If we take the underived sheaf hom, $\underline{\mathrm{Ext}}^0(k(x), \mathcal{O}_X)$ we get nothing. Indeed, if I is a maximal ideal of a domain R, then $\mathrm{Hom}_R(R/I, R) = 0$ because 1 has no torsion. So locally $\underline{\mathrm{Ext}}^0(k(x), \mathcal{O}_X)$ is zero and therefore globally. However, once we take *derived* sheaf Hom, we get an interesting complex.

13.19.3. A very special case. — Let's see a special case. Assume $Y = D \subset X$ is a *divisor*, i.e. it's of codimension $c = 1$. Since X is smooth, this is equivalent to its ideal sheaf I_D being a line bundle (i.e. D is locally cut out by one equation). In this case, we often write $I_D = \mathcal{O}_X(-D)$. Summarizing, we have a short exact sequence

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$$

where we used our shorthand $\mathcal{O}_D = i_*\mathcal{O}_D$. We write $\mathcal{O}_X(D)$ for the dual line bundle $\mathcal{O}_X(-D)^\vee$. We also write $F(D) = F \otimes \mathcal{O}(D)$.

***Theorem 13.28* (Adjunction formula).** — We have $\omega_D = i^*(\omega_X \otimes \mathcal{O}(D))$.

Corollary 13.29. — $(i_*\mathcal{O}_D)^\vee = i_*\mathcal{O}_D(D)[-1]$.

13.19.4. A less very special case. — Consider $Y \subset X$ of codimension $c > 1$ and let I_Y be its ideal sheaf. Then

$$H^k(I_Y^\vee) = \begin{cases} \mathcal{O}_Y & k = 0 \\ i_*\omega_Y \otimes \omega_Y^\vee & k = c - 1 \\ 0 & \text{otherwise} \end{cases}$$

Indeed, we have a short exact sequence

$$0 \to I_Y \to \mathcal{O}_X \to i_*\mathcal{O}_Y \to 0$$

Applying $(-)^\vee$ we get an exact triangle

$$\mathcal{O}_Y^\vee \to \mathcal{O}_X \to I_Y^\vee \to$$

and we simply stare at the long exact sequence in cohomology to obtain the result.

Proposition 13.30. — $\operatorname{supp} E^\vee = \operatorname{supp} E$

Proof. — Assume $E \otimes k(x) = 0$. Then $(E \otimes k(x))^\vee = 0$ We have

$$\begin{aligned} 0 = (E \otimes k(x))^\vee &= E^\vee \otimes k(x)^\vee \\ &= E^\vee \otimes i_*\omega_{\mathtt{pt}} \otimes \omega_X^\vee[-\dim X] \\ &= E^\vee \otimes i_*\mathbf{C} \otimes \omega_X^\vee[-\dim X] \\ &= E^\vee \otimes k(x)[-\dim X] \end{aligned}$$

hence $E^\vee \otimes k(x) = 0$. Since $E^{\vee\vee} = E$, the result follows. □

Excellent, now we have the tools to tackle Chapter 4 in Huybrechts's book.

14. Chapter 4: canonical

When we do not specify, all varieties are smooth and projective over $\mathbf{C}$. We say X, Y are *derived equivalent* if $\mathtt{D}(X) = \mathtt{D}(Y)$ as $\mathbf{C}$-linear triangulated categories.

Remark 14.1. — Let $f: X \to Y$ be an isomorphism of varieties. Then (underived) pullback $f^*: \mathtt{Coh}(Y) \to \mathtt{Coh}(X)$ is an equivalence. This is because, if $g = f^{-1}$, $g^*f^* = (fg)^* = \mathrm{id}^* = \mathrm{id}_{\mathtt{Coh}}$. Hence (derived) pullback $f^*: \mathtt{D}(Y) \to \mathtt{D}(X)$ is an equivalence.

Theorem 14.2. — Let $\Phi\colon \mathtt{D}(X) \to \mathtt{D}(Y)$ be an equivalence. Then $\Phi \circ S_X = S_Y \circ \Phi$.

Proof. — By Yoneda, we reduce to proving that

$$\mathrm{Hom}(E, \Phi S_X(F)) = \mathrm{Hom}(E, S_Y\Phi(F))$$

in a sufficiently functorial way. Let $\Psi = \Phi^{-1}$.

$$\begin{aligned}\mathrm{Hom}(E, S_Y\Phi(F)) &= \mathrm{Hom}(\Phi(F), E)^\vee \\ &= \mathrm{Hom}(F, \Psi(E))^\vee \\ &= \mathrm{Hom}(S_X^{-1}\Psi(E), F) \\ &= \mathrm{Hom}(E, \Phi S_X(F)).\end{aligned}$$

□

Slogan: equivalences commute with Serre functors.

Remark 14.3. — The proof above has nothing to do with varieties: it's about any triangulated category with a Serre functor.

Corollary 14.4. — If X and Y are derived equivalent then $\dim X = \dim Y$.

This is already an interesting result: dimension is a categorical invariant. You can't have a surface being derived equivalent to a fourfold.

Proof. — Let $\Phi\colon \mathtt{D}(X) \to \mathtt{D}(Y)$ be an equivalence. Then $\Phi S_X k(p) = S_Y \Phi k(p)$. But $\Phi S_X k(p) = \Phi k(p)[\dim X]$, while $S_Y\Phi k(p) = (\Phi k(p)) \otimes \omega_Y[\dim Y]$. Let $E = \Phi k(p)$, we have

$$E \otimes \omega_Y[\dim Y] = E \otimes [\dim X]$$

and since $(-) \otimes \omega_Y$ does not shift degrees we must have $\dim X = \dim Y$. □

OK, in the proof above we used at least three things without mention: $\omega \otimes k(p) = k(p)$ since $k(p)$ is a skyscraper; $\Phi(k(p)) \neq 0$ since $k(p) \neq 0$ and Φ is an equivalence; since $\otimes\omega_Y$ is exact it commutes with taking cohomology, i.e. $H^k(F \otimes L) = H^k(F) \otimes L$ for any line bundle L.

Corollary 14.5. — Suppose X and Y are derived equivalent. Then the orders of ω_X and ω_Y are the same.

For example tells you that a K3 surface can't be derived equivalent to $\mathbf{P}^1 \times C$ with C a curve.

Proof. — Let $d = \dim X = \dim Y$. Say $\omega_X^k = \mathcal{O}_X$. Then

$$[kd]\Phi = \Phi[kd] = \Phi S_X^k = S_Y^k \Phi = (-) \otimes \omega_Y^k[kd]\Phi$$

hence $\omega_Y^k = \mathcal{O}_Y$. By symmetry (i.e. using Φ^{-1}) we conclude. □

***Theorem 14.6* (Bondal–Orlov).** — Suppose X has ample (or anti-ample) canonical bundle. Suppose X and Y are derived equivalent. Then X and Y are isomorphic as varieties over $\mathbf{C}$.

Remark 14.7. — What's a cheap way to cook up varieties with ample or anti-ample canonical? Turns out it's pretty easy. Consider $X \subset \mathbf{P}^{N+1}$ a hypersurface of degree d. By adjunction, the canonical of X is $\omega_X = \mathcal{O}(-N-2+d)$. Thus, for $d > N+2$, ω_X is ample, for $d < N+2$ it's anti-ample, while for $d = N+2$ $\omega_X = \mathcal{O}_X$ is trivial (so Bondal-Orlov does not apply).

This Bondal-Orlov theorem is a derived analogue of Gabriel's theorem.

***Theorem 14.8* (Gabriel).** — Suppose $\mathtt{Coh}(X)$ is equivalent to $\mathtt{Coh}(Y)$, then X is isomorphic to Y.

Proof. — One way to prove this is the following. [For this proof, we suspend our convention that everything is derived] Skyscrapers can be characterized categorically. Indeed, let's call a sheaf F ***point-like*** if

- $\mathrm{Hom}(F, F) = \mathbf{C}$
- if $F \twoheadrightarrow G$ is a surjection, then either $G = 0$ or $F = G$.

Clearly skyscrapers are point-like. If F is point-like, then by the first condition $F \neq 0$ and thus there is $p \in \operatorname{supp} F$. Let $p \neq q \in \operatorname{supp} F$, let $i\colon \mathtt{pt} \to X$ be the inclusion of q. Then $F \twoheadrightarrow i_* i^* F$ is surjective (which is always the case for closed immersions). By the second condition, we have $F = i_* i^* F$ or $i_* i^* F = 0$. The first case would imply $\operatorname{supp} F = \{q\}$ which cannot be true, hence $i_* i^* F = 0$. From this we deduce that $\operatorname{supp} F = \{p\}$. The first condition will then imply that it must be a skyscraper.

That's great. Any equivalence $\mathtt{Coh}(X) \simeq \mathtt{Coh}(Y)$ then has to take point-like objects to point-like objects, i.e. skyscrapers to skyscrapers. Hence it will define a map $X \to Y$, which is forced to be an isomorphism. [To make this last part rigorous, one needs to define what *families* of points are. If S is a variety, then families of points in $\mathtt{Coh}(S \times X)$ correspond to graphs of morphisms $S \to X$. Doing this properly is a little delicate.] □

14.1. Ampleness. — Before proceeding with the Bondal–Orlov theorem, let's recall some basics of ample bundles.

Let V be a vector space. The variety $\mathbf{P}(V)$ is the prototype of a *moduli space*. The breakdown goes a little something like this.

- The *moduli problem*: we wish to parameterize lines (through the origin) in V.
- *Families* (aka the moduli functor): if S is a space, we must define what is a family of lines in V, parameterized by S.
- *Moduli space*: we find a variety $\mathbf{P}(V)$ which "represents" our moduli functor. More precisely: maps $S \to \mathbf{P}(V)$ correspond to families of lines in V, parameterized by S.

To recap, the variety $\mathbf{P}(V)$ is parameterizing lines in V. If $S \to \mathbf{P}(V)$, each $s \in S$ gives rise to a line $L_s \subset V$ which varies nicely as s varies in S. In particular, there is a *universal* family of lines on $\mathbf{P}(V)$ corresponding to the identity $\mathbf{P}(V) \to \mathbf{P}(V)$. We will see now that this universal family corresponds to the tautological bundle $\mathcal{O}(-1)$.

Recall that $\mathcal{O}(-1)$ is defined as the subset of $\mathbf{P}(V) \times V$ consisting of pairs (p, v) such that $v \in p$, when p is viewed as a line in V. The dual bundle is called $\mathcal{O}(1) = \underline{\mathrm{Hom}}(\mathcal{O}(-1), \mathcal{O})$. In general we define $\mathcal{O}(k) = \mathcal{O}(\pm 1)^{\otimes |k|}$ depending on whether k is positive or negative.

Recall the following fact about projective space.

- $H^0(\mathbf{P}(V), \mathcal{O}(k)) = \mathrm{Sym}^k V^*$, i.e. degree k homogeneous polynomial functions on V. Notice this is zero when $k < 0$.
- $H^i(\mathbf{P}(V), \mathcal{O}(k)) = 0$ for $0 < i < N = \dim \mathbf{P}(V)$
- $H^N(\mathbf{P}(V), \mathcal{O}(k)) \simeq (\mathrm{Sym}^{-k-N+1} V^*)^*$

where the last bulletpoint is Serre duality.

Theorem 14.9 **(Euler sequence)**. — We have a short exact sequence

$$0 \to \mathcal{O}(-1) \to \mathcal{O} \otimes_{\mathbf{C}} V \to T(-1) \to 0$$

where T is the tangent bundle of $\mathbf{P}(V)$. It follows that the canonical bundle of $\mathbf{P}(V)$ is $\omega_{\mathbf{P}(V)} = \mathcal{O}(-\dim V)$.

Let $f : X \to \mathbf{P}(V)$ be any map. Thus on X we have an exact triangle

$$f^*\mathcal{O}(-1) = \mathcal{O}_X(-1) \to \mathcal{O}_X \otimes_{\mathbf{C}} V \to f^*(T)(-1) \to$$

but since each term is a vector bundle (which are f^*-projective) it's actually a short exact sequence of sheaves. Call $V_X = \mathcal{O}_X \otimes_{\mathbf{C}} V$. We claim that maps $X \to \mathbf{P}(V)$ are actually the same thing as rank one sub-bundles of V_X, i.e. inclusions $L \hookrightarrow V_X$ where the quotient V_X/L is locally free.

Indeed, suppose you have such a sub-bundle $L \hookrightarrow V_X$, how do we define $f : X \to \mathbf{P}(V)$? Well, if $x \in X$, the fibre $L_p \subset V$ is a line in V, hence a point of $\mathbf{P}(V)$! So $f : X \to \mathbf{P}(V)$ sends x to L_x. The condition that V_X/L is locally free makes sure that $L_x \hookrightarrow V$ is always injective. It's easy to check then that $f^*\mathcal{O}(-1) = L$.

Remark 14.10. — Recall that we have an injection of sheaves $\mathcal{O}(-1) \hookrightarrow \mathcal{O}$ on $\mathbf{P}^1$. However, the quotient is $k(p)$, for some $p \in \mathbf{P}^1$. Indeed, the map $\mathcal{O}(-1)_x \to \mathbf{C}$ on fibres is an isomorphism everywhere, except at p where it's the zero map.

Let's go back to moduli problems. If S is a variety, we define an S-family of lines in V to be a line bundle L on S together with an inclusion $L \hookrightarrow V_S = V \otimes_{\mathbf{C}} \mathcal{O}_S$ such that the quotient V_S/L is locally free. The discussion above (with X replaced by S) shows that any family of lines gives rise to a map $S \to \mathbf{P}(V)$. Conversely, a map $S \to \mathbf{P}(V)$ gives rise to a family of lines by pulling back $\mathcal{O}(-1)$. Hence, we've just shown that $\mathbf{P}(V)$ is truly the moduli space of lines in V and that $\mathcal{O}(-1)$ is the universal family of lines.

14.1.1. Dualize. — The dual point of view is also useful. A sub-bundle $L \subset V_X$ corresponds to a surjection $V_X^\vee \twoheadrightarrow L^\vee$. So, we may describe maps to $\mathbf{P}(V)$ as quotients $V_X^\vee \twoheadrightarrow L^\vee$, with L a line bundle. Dually to before, $f^*\mathcal{O}(1) = L^\vee$.

14.1.2. Globally generated. — So far, we've seen that if L is a line bundle on X, if we have an inclusion $L \hookrightarrow V_X$ then we induce a map $f : X \to \mathbf{P}(V)$, such that $L = f^*\mathcal{O}(-1)$. Dually, if we have a surjection $V_X \twoheadrightarrow L$ we get a map $g : X \to \mathbf{P}(V^\vee)$ such that $L = g^*\mathcal{O}(1)$. So the following question is natural: if L is a line bundle on X, when can we find an inclusion $L \subset V_X$ or a surjection $V_X \twoheadrightarrow L^\vee$?

Traditionally, the notation is dual to ours: so that one asks when there is a surjection $V_X \twoheadrightarrow L$.

Let, as usual, $\gamma\colon X \to \mathtt{pt}$. Note that $V_X = \gamma^*(V)$. Thus

$$\mathrm{Hom}(V_X, L) = \mathrm{Hom}(V, R^0\gamma_* L) = \mathrm{Hom}(V, H^0(X, L)).$$

In other words, we are looking for linear maps $V \to H^0(X, L)$.

In particular, $V_X \to L$ factors through as $V_X \to H^0(X, L) \otimes_{\mathbf{C}} \mathcal{O}_X \to L$ thus if $V_X \twoheadrightarrow L$ is surjective then $H^0(X, L) \otimes_{\mathbf{C}} \mathcal{O}_X \twoheadrightarrow L$ must also be surjective.

Definition 14.11. — We say L is *globally generated* if the natural evaluation map $H^0(X, L) \otimes_{\mathbf{C}} \mathcal{O}_X \to L$ is surjective.

If $V = H^0(X, L)^\vee$, a globally generated line bundle always gives rise to a map $X \to \mathbf{P}(V)$.

14.2. Ample. — Here's an example of a globally generated line bundle: $L = \mathcal{O}_X$. However, if X is proper (and integral) then $H^0(X, \mathcal{O}_X) = \mathbf{C}$. So the induced map is $X \to \mathbf{P}^0 = \mathtt{pt}$. Not very interesting. Ampleness comes in precisely to rule out this case.

Definition 14.12. — A line bundle L on X is ***very ample*** if there is an inclusion $i\colon X \hookrightarrow \mathbf{P}(V)$ such that $L = i^*\mathcal{O}(1)$. We say L is ***ample*** if there exists $k \geq 0$ such that $L^{\otimes k}$ is very ample. Finally, we say L is ***anti-ample*** if $L^\vee$ is ample.

14.2.1. Hilbert. — Let $E, F \in \mathtt{D}(X)$, we define the *Euler characteristic* to be

$$\chi(E, F) = \sum_i \dim_{\mathbf{C}} \mathrm{Ext}^i(E, F).$$

Digression 14.13. — We can rephrase this in a more pretentious way as follows. If $W \in \mathtt{D}(\mathbf{C})$ is a (bounded) complex of (finite dimensional) vector spaces, we define its ***dimension*** to be

$$\dim W = \sum_i (-1)^i \dim W^i = \sum_i (-1)^i \dim H^i(W).$$

If $E, F \in \mathtt{D}(X)$ and $\gamma\colon X \to \mathtt{pt}$ is as usual the map to a point, we have

$$\chi(E, F) = \dim \mathrm{RHom}_X(E, F) = \dim \gamma_* \underline{\mathrm{Hom}}(E, F).$$

As a special case, we have

$$\chi(X, F) = \chi(\mathcal{O}_X, F) = \sum_i (-1)^i \dim H^i(X, F)$$

which is sometimes called the ***holomorphic*** Euler characteristic (to distinguish it from the topological Euler characteristic of the topological space X).

Let us fix $L \in \mathtt{D}(X)$. We call $E(n) = E \otimes L^{\otimes n}$. We define the *Hilbert function* of E to be $H_E(k) = \chi(X, E(n))$. The Hilbert function obviously depends on L. When L is an ample line bundle, this function $H_E(k)$ is actually ***polynomial*** and the corresponding polynomial is called the *Hilbert polynomial* of E. The coeffiecints of this polynomial contain information about E. We will content ourselves with the following fact.

Proposition 14.14. — Let $F \in \mathtt{Coh}(X)$ and let L be ample. Then $\deg H_F = \dim \operatorname{supp} F$.

14.3. Proof of Bondal-Orlov. — We will mimic the proof of Gabriel's theorem, namely we want to characterize skyscrapers categorically. Recall that $S = (-) \otimes \omega_X[\dim X]$ denotes the Serre functor, which is intrinsic to the category $D(X)$. We say an object $E \in D(X)$ is *point-like* if

- $\mathrm{Hom}(E, E) = \mathbf{C}$.
- $\mathrm{Hom}(E, E[i]) = 0$ for $i < 0$.
- There exists $r \in \mathbf{Z}$ and an isomorphism $S(E) \cong E[r]$.

Clearly, if $E = k(p)$ is a skyscraper, then it's pointlike. This is because $S(k(p)) = k(p) \otimes \omega_X[\dim X] = k(p)[\dim X]$. Same goes for $k(p)[j]$ for any j.

It's also clear that point-like objects are perserved under equivalences (since Serre functors and shifts commute with those). What we need to show is that indeed, a point-like object is always a skyscraper (up to maybe a shift). This is *not* true in general. Indeed, suppose X is an elliptic curve. Then $S_X = [1]$. If L is any line bundle, we have

- $\mathrm{Hom}(L, L) = \mathrm{Hom}(\mathcal{O}, \mathcal{O}) = \mathbf{C}$
- $\mathrm{Hom}(L, L[i]) = 0$ for $i < 0$
- $S(L) = L[1]$.

So in this case point-like objects can be very far from being actual points.

Remark 14.15. — In general we call a category T with Serre functor $S_T = [d]$ *Calabi-Yau* of dimension d. Or we just write CY_d. If X is a smooth and projective variety has trivial canonical bundle $\omega_X = \mathcal{O}_X$, then $D(X)$ is $CY_{\dim X}$. The converse is not true.

Going back to the theorem, suppose X now has *ample* canonical bundle. Let E be point-like. Suppose actually $E \in \mathrm{Coh}(X)$ is a sheaf. The condition $S(E) = E[r]$ for some r means $E \otimes \omega_X[\dim X] = E[r]$, which implies $r = \dim X$. More importantly, we see that $E \otimes \omega_X = E$. Consider now the Hilbert polynomial H_E of E. Since the value $H_E(k)$ is constant, its degree must be zero. In particular, $\dim \operatorname{supp} E = 0$. The condition $\mathrm{Hom}(E, E) = \mathbf{C}$ then lets us conclude that E is $k(p)$ for some p.

For a general complex E, one must work a little harder. The same argument implies that each cohomology sheaf $H^i(E)$ is supported in dimension zero. Moreover, if it were supported in more than one point we'd violate $\mathrm{Hom}(E, E) = \mathbf{C}$, so E must be supported at a single point. Suppose $H^a(E) \neq 0 \neq H^b(E)$ for a minimal and b maximal. Since $H^a(E), H^b(E)$ are modules supported at the maximal ideal of a local artinian ring, there exists a non-zero map $H^b(E) \to k(p) \to H^a(E)$. Thus we have a non-zero map

$$E[b] \to H^b(E) \to H^a(E) \to E[a]$$

and since $a - b < 0$, this contradicts $\mathrm{Hom}(E, E[i]) = 0$ for $i < 0$.

This shows that when ω_X is ample, any point-like object is actually the shift of a skyscraper. The case where $\omega_X^\vee$ is similar, by using the inverse of the Serre functor.

To prove the Bondal-Orlov theorem, we may argue as follows. We only sketch the details. By being a little careful, we can define a moduli space of point-like objects of $D(X)$. Call this P_X. In general we have an embedding $X \to P_X$. If $D(X) \simeq D(Y)$ is an equivalence, then point-like objects are sent to point-like objects, so $P_X \simeq P_Y$. If we assume ω_X is ample or anti-ample, then (up to modding out by the action of the shifts) $X \to P_X$ is an isomorphism.

So we look at the map in the other way $Y \to P_Y \simeq P_X \simeq X$. Then we check that $Y \to X$ is both an open and a closed immersion, which implies it's an isomorphism.

14.4. Autoequivalences. — Gabriel's theorem actually proves more, it gives a characterization of the group of auto-equivalences of $\mathtt{Coh}(X)$. There are two obvious sources of auto-equivalences: automorphisms and tensoring with line bundles. Gabriel says that actually these are all!

$$\mathrm{Aut}(\mathtt{Coh}(X)) = \mathrm{Pic}(X) \ltimes \mathrm{Aut}(X)$$

where if $f\colon X \to X$ is an automorphism we consider the autoequivalence $f_* = (f^*)^{-1}$ (because pullback is contravariant).

Bondal-Orlov shows the analogous result: if X has ample or anti-ample canonical bundle then

$$\mathrm{Aut}(\mathtt{D}(X)) = \mathbf{Z} \times \mathrm{Aut}(\mathtt{Coh}(X))$$

where $\mathbf{Z}$ acts via shifts.

15. Chapter 5: kernels

Consider $p\colon X \times Y$, $q\colon X \times Y \to Y$. If $K \in \mathtt{D}(X \times Y)$ we define the *integral transform* with kernel K to be the functor $\Phi_K\colon \mathtt{D}(X) \to \mathtt{D}(Y)$ defined as

$$\Phi_K(E) = q_*(K \otimes p^* E).$$

Example 15.1. — The composition of integral transforms is an integral transform. If Φ_K, Φ_L are transforms, then the kernel defining the composition $\Phi_K \circ \Phi_L$ is denoted by $K * L$ and called the *convolution* of kernels.

When an integral transform is an equivalence, we say it's a *Fourier–Mukai transform.* We also say X and Y are *Fourier–Mukai partners.*

If $f\colon X \to Y$ is any map, then we have a morphism $\Gamma\colon X \to X \times Y$ sending x to $(x, f(x))$. Let $K = \Gamma_* \mathcal{O}_X$, then $\Phi_K = f_*$.

Example 15.2. — In the case where $f = \mathrm{id}$, we typically call $\Gamma = \Delta$. If $E \in \mathtt{D}(X)$, then $\Phi_{\Delta_* E}(F) = E \otimes F$.

Suppose $g\colon Y \to X$. We have $\Lambda\colon Y \to X \times Y$ sending y to $(g(y), y)$. Then $\Phi_\Lambda = g^*$.

Convention: suppose $K \in \mathtt{D}(X \times Y)$. We will write Φ_K for $q_*(K \otimes p^*(-))$ and $\Psi_K = p_*(K \otimes q^*(-))$.

Proposition 15.3. — Let $\Phi = \Phi_K$ be an integral transform. Define

$$\Phi^L = \Psi_{K^\vee} \circ S_Y$$
$$\Phi^R = S_X \circ \Psi_{K^\vee}$$

Then Φ^L is the left adjoint of Φ and Φ^R is the right adjoint of Φ.

Define

$$K_L = K^\vee \otimes q^*\omega_Y[\dim Y]$$
$$K_R = K^\vee \otimes p^*\omega_X[\dim X]$$

we have

$$\Phi^L = \Psi_{K_L} \qquad \Phi^R = \Psi_{K_R}.$$

***Theorem 15.4* (Orlov).** — Let $F\colon D(X) \to D(Y)$ be a fully faithful exact functor. Then there exists K (unique up to isomorhism) and an isomorphism of functors $F \simeq \Phi_K$.

Remark 15.5. — If one is willing to use enhancements (i.e. consider derived categories as dg or ∞-categories) then *any* (dg or ∞) functor is an integral transform.

This suggests that the category of (exact, $\mathbf{C}$-linear) functors $\mathrm{Fun}(D(X), D(Y))$ should be equivalent to $D(X \times Y)$. This, however, is not the case.

Remark 15.6. — Let C be an elliptic curve, i.e. $\omega_C = \mathcal{O}_C$ is trivial. Let $\mathcal{O}_\Delta$ be the structure sheaf of the diagonal in $C \times C$. Since $C \times C$ is a surface and $\omega_{C\times C} = \mathcal{O}_{C\times C}$, we have $\mathrm{Ext}^2(\mathcal{O}_\Delta, \mathcal{O}_\Delta) = \mathrm{Hom}(\mathcal{O}_\Delta, \mathcal{O}_\Delta) = \mathrm{Hom}_C(\mathcal{O}_C, \mathcal{O}_C) = \mathbf{C} \neq 0$. This means there is a non-zero map $\tau\colon \mathcal{O}_\Delta \to \mathcal{O}_\Delta[2]$ in $D(C \times C)$.

On the other hand, consider the induced natural transformation $\Phi_\tau\colon \mathrm{id} \to \mathrm{id}[2]$, where we just used the fact that $\Phi_{\mathcal{O}_\Delta} = \mathrm{id}$. If $E \in D(C)$, then $\Phi_\tau(E)\colon E \to E[2]$ is an element of $\mathrm{Ext}^2(E, E)$.

If $E \in \mathrm{Coh}(C)$, then $\Phi_\tau(E) = 0$ as $\dim C = 1$. In general, we have seen that any $E = \bigoplus F_i[i]$ with $F_i \in \mathrm{Coh}(C)$. This implies that $\Phi_\tau(E)$ is the zero morphism for any E.

What have we learned? Natural transformations between functors do not correspond to morphisms between kernels. As usual, to fix this one needs enhancements.

Proposition 15.7. — Let $K_i \in D(X_i \times Y_i)$ be kernels inducing equivalences $\Phi_{K_i}\colon D(X_i) \xrightarrow{\sim} D(Y_i)$, for $i = 1, 2$. Then the box product $K_1 \boxtimes K_2$ induces an equivalence $\Phi_{K_1\boxtimes K_2}\colon D(X_1 \times X_2) \to D(Y_1 \times Y_2)$.

Recall that the box product is defined by tensoring the pullbacks of the two objects from each factor. Concretely, if $\pi_i\colon (X_1 \times X_2) \times (Y_1 \times Y_2) \to X_i \times Y_i$ denotes the projection, for $i = 1, 2$, then $K_1 \boxtimes K_2 = \pi_1^*(K_1) \otimes \pi_2^*(K_2)$.

Proof. — Recall that the composition of integral transforms is an integral transform, via convolution of kernels. Let L_i denote the kernel corresponding to the inverse of Φ_{K_i}. We know $K_i * L_i = \mathcal{O}_\Delta$ and $L_i * K_i = \mathcal{O}_\Delta$, where we are abusing the Δ notation.

To check $K_1\boxtimes K_2$ induces an equivalence, it suffices to check that $(K_1\boxtimes K_2)*(L_1\boxtimes L_2) = \mathcal{O}_\Delta$, and similarly for the opposite convolution. Exercise in base change. □

15.1. Grothendieck group. — We now define the group $K(X) = K(\mathrm{Coh}(X))$. It is the free abelian group spanned by isomorphism classes of coherent sheaves where, for each short exact sequence

$$0 \to A \to B \to C \to 0$$

we have the relation $[B] = [A] + [C]$. Notice that the class $[0]$ is the zero element, and moreover $[A \oplus B] = [A] + [B]$.

Remark 15.8. — This definition makes sense for any abelian category.

Digression 15.9. — $K(X)$ should actually be denoted by $K_0(X)$ as there are *higher* K-groups. We will not consider those here (they are basically impossible to compute).

This definition can actually be extended to $\mathtt{D}(X)$. Define the Grothendieck group K' of $\mathtt{D}(X)$ to be the free abelian group spanned by isomorphism classes of complexes where, for each exact triangle

$$E \to F \to G \xrightarrow{+}$$

we have $[F] = [E] + [G]$.

Remark 15.10. — As before, this definition makes sense for any triangulated category.

Proposition 15.11. — The two Grothendieck groups we just defined actually coincide.

Proof. — Indeed, if $A \in \mathtt{Coh}(X)$ we can view it as a complex sitting in degree zero. A short exact sequence becomes an exact triangle, so we have a well defined group homomorphism $K(X) \to K'$. On the other hand, given $E \in \mathtt{D}(X)$ we always have (using truncations)

$$[E] = \sum_i (-1)^i [H^i(E)] = \sum_i (-1)^i [E^i]$$

in K' (the last equality is satisfied only if E is being represented by a *bounded* complex $E^\bullet$ of *coherent* sheaves). Hence the map $K(X) \to K'$ is bijective. □

We'll simply forget about the notation K' from now on (it's a made up notation anyway) and just write $K(X)$ to mean either.

Since X is always assumed to be smooth and projective, the tensor product $E \otimes F$ of two complexes still lives in $\mathtt{D}(X)$, i.e. it stays bounded (this was because any sheaf has a bounded resolution by vector bundles). Hence, $K(X)$ becomes a ring under (derived) tensor product.

Remark 15.12. — Notice that it's important the tensor product is derived, because otherwise it wouldn't be compatible with the defining equivalence relation of $K(X)$. Recall that, if $A, B \in \mathtt{Coh}(X)$, we write $\mathrm{Tor}_i(A, B) = H^{-i}(A \otimes B)$.

The product structure on $K(X)$ is then

$$[A] \cdot [B] = [A \otimes B] = \sum_i (-1)^i [\mathrm{Tor}_i(A, B)].$$

Similarly to $\otimes$, any (exact) functor $\Phi \colon \mathtt{D}(X) \to \mathtt{D}(Y)$ will induce a group homomorphism $K(X) \to K(Y)$ by taking $[E]$ to $[\Phi(E)]$. In particular, for $f \colon X \to Y$ we have

$$\begin{aligned} f_* \colon K(X) &\to K(Y) \\ [E] &\mapsto [f_* E] \end{aligned}$$

if $A \in \mathtt{Coh}(X)$ we have $[f_* A] = \sum_i (-1)^i [R^i f_* A]$. Justifying this last line actually needs some work (but it's easy once we are allowed to use spectral sequences).

Notice that $f^*\colon K(Y) \to K(X)$ is not only a group hom, but also a *ring* homomorphism. Since $f^*(E \otimes F) = f^*E \otimes f^*F$.

Remark 15.13. — Notice that $[E[k]] = (-1)^k[E]$.

Euler. — The Grothendieck group comes with a bilinear map

$$\chi\colon K(X) \otimes K(X) \to \mathbf{Z}$$
$$\chi(E, F) = \sum_i (-1)^i \dim \operatorname{Ext}^i(E, F) = \sum_i (-1)^i \dim \operatorname{Hom}(E, F[i])$$

called the *Euler form.*

Proposition 15.14. — Even though χ is not symmetric, its left and right radicals coincide.

Proof. — Indeed, suppose E is such that $\chi(E, F) = 0$ for all $F \in D(X)$. By Serre duality, $\chi(G, E) = (-1)^n \chi(E, G \otimes \omega_X)$ where $n = \dim X$. The rest follows. □

The quotient of K(X) by its radical is sometimes called $K_{\mathrm{num}}(X)$ the *numerical* Grothendieck group.

Digression 15.15. — We will mention later that the chern character provides an *ungraded* isomorphism between $K(X) \otimes \mathbf{Q}$ and $CH(X) \otimes \mathbf{Q}$ the Chow groups of X tensored with $\mathbf{Q}$. The Chow groups are defined to be algebraic cycles modulo rational equivalence. In a nutshell, we declare two subvarieties $Y, Y' \subset X$ rationally equivalent if there is a subvariety $Z \subset X \times \mathbf{P}^1$, flat over $\mathbf{P}^1$, such that $Z \times \{0\} = Y, Z \times \{\infty\} = Y'$. In general understanding even $CH^0(X)$, the group of zero-cycles, is highly non-trivial.

We say two cycles α, α' are *numerically equivalent* $\alpha \cong \alpha'$ if for any other cycle β the intersection product $\alpha \cdot \beta = \alpha' \cdot \beta$ coincide. One defines N(X) to be the quotient of CH(X) modulo numerical equivalence.

One shows that the chern character descends to an isomorphism between $K_{\mathrm{num}}(X) \otimes \mathbf{Q}$ and $N(X) \otimes \mathbf{Q}$.

15.1.1. Integral transforms on K. — If $K \in D(X \times Y)$ then $[K] \in K(X \times Y)$ is a class in the Grothendieck group. The integral transform $\Phi_K\colon D(X) \to D(Y)$ induces a group homomorphism $K(X) \to K(Y)$ defined by the same formula: the class [F] is sent to $q_*([K] \cdot [p^*F])$.

Diagrammatically, we write

$$\begin{array}{ccc} D(X) & \xrightarrow{\Phi} & D(Y) \\ \downarrow{[\,]} & & \downarrow{[\,]} \\ K(X) & \longrightarrow & K(Y) \end{array}$$

Similarly, if we had a cohomology class $e \in H^*(X \times Y)$ we could do the following. If $\alpha \in H^*(X)$ we can pull it back $p^*\alpha$, cup it with e (which is Poincaré dual to *intersecting*) and push forward to Y. By pushforward here we mean the following simple thing. Cohomology does not come with pushforward maps, but homology does. So we have $q_*\colon H_*(X \times Y) \to H_*(Y)$. However, since our varieties are smooth and projective over $\mathbf{C}$ they are compact complex manifolds. In particular, they come with a canonical orientation and hence a

Poincaré duality isomorphism PD: $H_k(X) \xrightarrow{\sim} H^{2n-k}(X)$, where $n = \dim X$ as a complex manifold. Hence we may define $q_*: H^*(X \times Y) \to H^*(Y)$ by applying PD then q_* in homology then PD again. I guess for this to be rigorous we should interpret $H^*(X)$ as singular cohomology with rational coefficients. We should note that, while p^* preserves the grading in homology, q_* shifts it. In general, if $f: X \to Y$ is a map of varieties, f_* will send $H^i(X)$ to $H^{i+2\dim Y - 2\dim X}(Y)$.

In conclusion, given $e \in H^*(X \times Y)$ we may define a cohomological integral transform $\Phi_e: H^*(X) \to H^*(Y)$ sending α to $q_*(e \cup p^*\alpha)$. Note that if e does not live in a single degree, the transform Φ_e might heavily mess up the grading in cohomology.

Starting with an integral transform between derived categories, we know how to obtain a transform between the K-groups. How can we obtain one between cohomology groups? We would need a way to pass from classes [E] to singular cohomology. Thankfully the Chern character does the trick.

15.2. Chern classes. — Let E be a vector bundle on X. To E we may attach cohomology classes $c_k(E) \in H^{2k}(X, \mathbf{Z})$ called *chern classes*. These classes satisfy (and are uniquely determined by a subset of) the following.

1. $c_0(E) = 1$
2. $f^* c_k(E) = c_k(f^*E)$
3. for any short exact sequence $0 \to E \to G \to F \to 0$, $c_k(G) = \sum_{i+j=k} c_i(E) \cup c_j(F)$
4. on $\mathbf{CP}^N$, $c_1(\mathcal{O}(-1)) = -t$ where t is the standard generator of the ring $H^*(\mathbf{CP}^N, \mathbf{Z})$. In other words t is Poincaré dual to the class of the hyperplane $\mathbf{CP}^{N-1} \subset \mathbf{CP}^N$.
5. if rank of E is r then $c_k(E) = 0$ for $k > r$.

For cohomology classes, we sometimes write $\alpha\beta$ for $\alpha \cup \beta$.

Remark 15.16. — For example from the second axiom it follows that, for any trivial bundle $V \otimes_{\mathbf{C}} \mathcal{O}$ on any X, we have $c_k(V \otimes_{\mathbf{C}} \mathcal{O}) = 0$ for any $k > 0$.

Remark 15.17. — Suppose E has a nowhere vanishing section. In other words an injective map $\mathcal{O} \to E$ with locally free quotient. It follows that $c_r(E) = 0$, where r is the rank of E.

More generally, if $\mathcal{O}^{\oplus k}$ injects in E with locally free quotient, we have $0 = c_r(E) = c_{r-1}(E) = \cdots = c_{r-k+1}$.

So we see a relationship between having linearly independent sections and vanishing chern classes.

What we actually want is the chern *character*. The chern character ch(E) of a vector bundle E is a cohomology class $\mathrm{ch}(E) \in H^*(X, \mathbf{Q})$, hence it is a sum $\mathrm{ch}(E) = \sum_k \mathrm{ch}_k(E)$ where $\mathrm{ch}_k(E) \in H^{2k}(X, \mathbf{Q})$. It satisfies the following.

1. $\mathrm{ch}_0(E) = \mathrm{rk}(E)$
2. $\mathrm{ch}_1(E) = c_1(E)$
3. for any short exact sequence $0 \to E \to F \to G \to 0$ we have $\mathrm{ch}(F) = \mathrm{ch}(E) + \mathrm{ch}(G)$
4. $\mathrm{ch}(E \otimes G) = \mathrm{ch}(E) \cup \mathrm{ch}(G)$.
5. $\mathrm{ch}_i(E^\vee) = (-1)^i \mathrm{ch}_i(E)$

To define it, we first look at line bundles. For L a line bundle, we define

$$\mathrm{ch}(L) = \exp(c_1(L))$$

where exp is defined as usual as a formal power series (but since powers of cohomology classes vanish after a while, this is really only a polynomial, so it's legit).

Remark 15.18. — It's because of the exp factor (and all the denominators it comes with) that we had to switch from $H^*(X, \mathbf{Z})$ to $H^*(X, \mathbf{C})$.

To define the chern character for a general vector bundle, one proceeds as follows. Let E be a vector bundle on X. Let Y be the *relative flag variety* of E, which parameterizes pairs (x, F) where $x \in X$ and F is a full flag of the fibre E_x (full flags are maximal sequences $F_0 \subset F_1 \subset F_2 \subset \cdots E_x$ where each quotient E_i/E_{i-1} is one-dimensional). There is an obvious map $p\colon Y \to X$ by forgetting the flag.

It turns out that $p^*\colon H^*(X, \mathbf{Q}) \to H^*(Y, \mathbf{Q})$ is injective (same if we were doing this for Chow groups). Tautologically, p^*E comes with a filtration where all quotients are line bundles. By enforcing additivity on short exact sequence we may then define ch(E) for any vector bundle.

Grothendieck again. — OK, what does this have to do with the Grothendieck group? Well, ch is additive on exact sequences of vector bundles. Any coherent sheaf can be resolved by finitely many vector bundles hence we may define

$$\mathrm{ch}(F) = \sum_i (-1)^i \,\mathrm{ch}(E^i)$$

for any resolution $E^\bullet$ of F with E^i vector bundles. Using this, one immediately has a well-defined *ring* homomorphism

$$\mathrm{ch}\colon K(X) \to H^*(X, \mathbf{Q})$$

Remark 15.19. — This can actually be refined, by replacing $H^*(X, \mathbf{Q})$ with the Chow group $CH^*(X) \otimes \mathbf{Q}$. In other words, chern classes can be refined to produce not just a singular cohomology class but actually algebraic cycles (up to rational equivalence). In particular, chern character are defined over *any* field. One can also show that ch is an isomorphism up to torsion. In other words, $K(X) \otimes \mathbf{Q} \simeq CH^*(X) \otimes \mathbf{Q}$ via ch.

Since $K(X) = K(\mathtt{Coh}(X)) = K(\mathtt{D}(X))$ one may define ch(E) for any $E \in \mathtt{D}(X)$.

Todd. — One natural question arises. Suppose $f\colon X \to Y$ is a map (as always, everything smooth and projective). We always have $f^* \mathrm{ch}(E) = \mathrm{ch}(f^*E)$. In other words, the following diagram commutes.

$$\begin{array}{ccc} K(Y) & \xrightarrow{\mathrm{ch}} & H^*(Y, \mathbf{Q}) \\ \downarrow{f^*} & & \downarrow{f^*} \\ K(X) & \xrightarrow{\mathrm{ch}} & H^*(X, \mathbf{Q}) \end{array}$$

What happens with pushforward? Turns out that in general $f_* \mathrm{ch}(E) \neq \mathrm{ch}(f_*E)$. Nevertheless, Riemann-Roch comes to the rescue! For this, we need to introduce the Todd class.

Similarly to ch we define it first for line bundles

$$\mathrm{td}(L) = \frac{c_1(L)}{1 - \exp(-c_1(L))}$$

and then extend it to all vector bundles.

The Todd class is multiplicative, meaning that $\mathrm{td}(E_1 \oplus E_2) = \mathrm{td}(E_1)\,\mathrm{td}(E_2)$.

Remark 15.20. — In my experience, what one does in practice is remember the formal properties of characteristic classes and look up the explicit formula in the back of Hartshorne whenever one needs to compute anything. For example,

$$\mathrm{ch} = \mathrm{rk} + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3) + \cdots$$
$$\mathrm{td} = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 \cdots$$

By the *Todd class of* X one means $\mathrm{td}(X) = \mathrm{td}(T_X)$ the Todd class of its tangent bundle.

***Theorem 15.21* (Grothendieck-Riemann-Roch).** — For any $\alpha \in K(X)$, we have

$$\mathrm{ch}(f_*\alpha)\,\mathrm{td}(Y) = f_*(\mathrm{ch}(\alpha)\,\mathrm{td}(X))$$

In other words, td(X) is the fudge factor that makes the diagram we want commute.

$$\begin{array}{ccc} K(X) & \xrightarrow{\mathrm{ch}(-)\,\mathrm{td}(X)} & H^*(X, \mathbf{Q}) \\ \downarrow f_* & & \downarrow f_* \\ K(Y) & \xrightarrow{\mathrm{ch}(-)\,\mathrm{td}(Y)} & H^*(Y, \mathbf{Q}) \end{array}$$

In the special case of $f\colon X \to \mathtt{pt}$, we obtain what is classically called Hirzebruch-Riemann-Roch. In this case, $f_*\alpha$ is typically denoted by $\int_X \alpha$, as it's really integrating a form against the fundamental class.

***Theorem 15.22* (HRR).** — For any $e \in K(X)$

$$\chi(e) = \int_X \mathrm{ch}(e)\,\mathrm{td}(X)$$

which is a special case of

$$\chi(E, F) = \int_X \mathrm{ch}(E^\vee)\,\mathrm{ch}(F)\,\mathrm{td}(X)$$

(it's the case where $E = \mathcal{O}_X$)

When $\dim X = 1$ we obtain the classical Riemann-Roch. Let's see how. If C is a curve, then $H^k(C) = 0$ for $k > 2$. So we can only have c_0 and c_1 classes. Moreover, classically one calls the *degree* of a vector bundle the quantity $\deg(E) = \int_C c_1(E)$.

Since C is a curve, the Todd class is also pretty easy:

$$\mathrm{td}(C) = 1 + \frac{1}{2}c_1(T_C) = 1 - \frac{1}{2}c_1(\omega_C)$$

One need a basic fact from topology. Let X be a complex manifold of dimension n.

$$\int_X c_n(T_X) = \chi(X)$$

in the case of a curve this becomes

$$\int_C c_1(T_C) = 2 - 2g$$

where g is the genus of the curve.

***Theorem 15.23* (RR).** — Let C be a curve. Let L be a line bundle on C. HRR tells us

$$\chi(L) = \int_C \mathrm{ch}(L)\,\mathrm{td}(C) = \deg(L) + \frac{1}{2}\deg(T_C) = \deg(L) + 1 - g.$$

On the other hand,

$$\chi(L) = \dim H^0(C, L) - \dim H^1(C, L) = \dim H^0(C, L) - \dim H^0(C, L^\vee \otimes \omega_C).$$

So if we interpret L as giving be a divisor class D, this becomes $l(D) - l(K_C - D)$ where K_C is the divisor class corresponding to ω_C. We may rewrite

$$l(D) - l(K_C - D) = \deg(D) + 1 - g$$

which is how the theorem is classically stated.

15.3. Mukai vectors. — To obtain compatibility with integral transforms we need one last bit of definition. We define the *Mukai vector* of E to be

$$v(E) = \mathrm{ch}(E)\sqrt{\mathrm{td}(X)}$$

The square root exists as td is of the form $1 + \alpha$ with α in degrees bigger than zero.

With all this in place, Riemann-Roch tells us that the following diagram commutes for any $e \in K(X \times Y)$.

$$\begin{array}{ccc} K(X) & \xrightarrow{\Phi_e} & K(Y) \\ \big\downarrow v & & \big\downarrow v \\ H^*(X, \mathbf{Q}) & \xrightarrow{\Phi_{v(e)}} & H^*(Y, \mathbf{Q}) \end{array}$$

where we wrote Φ_e for the integral transform at the level of the K-group with kernel e and $\Phi_{v(e)}$ for the integral transform at the level of cohomology with kernel $v(e)$.

Proposition 15.24. — In particular, let $P \in D(X \times Y)$. Then, for any $E \in D(X)$

$$q_*(v(P)p^*(v(E))) = v\left(q_*(P \otimes p^*E)\right)$$

Remark 15.25. — Notice that the cohomological integral transform $\Phi^H_{v(P)} \colon H^*(X, \mathbf{Q}) \to H^*(Y, \mathbf{Q})$ is $\mathbf{Q}$-linear but *not* a ring homomorphism. Moreover, it does *not* respect the grading (not even up to a shift). However, since ch and td are *even*, Φ^H respects the parity, in the sense that $\bigoplus_k H^{2k}(X)$ is sent to $\bigoplus_k H^{2k}(Y)$ and $\bigoplus_k H^{2k+1}(X)$ is sent to $\bigoplus_k H^{2k+1}(Y)$.

Proposition 15.26. — Suppose $\Phi_P\colon \mathtt{D}(X) \to \mathtt{D}(Y)$, $\Phi_Q \circ \mathtt{D}(Y) \to \mathtt{D}(Z)$ are two integral transforms and let $\Phi_R = \Phi_Q\colon \Phi_P$ be the composition. Write Φ^H for the associated map on H^* by taking Mukai vectors everywhere. Then $\Phi^H_{v(R)} = \Phi^H_{v(Q)} \circ \Phi^H_{v(P)}$.

Proposition 15.27. — Suppose Φ_P is an equivalence, then $\Phi^H_{v(P)}$ is an isomorphism of $\mathbf{Q}$-vector spaces.

Remark 15.28. — This statement is actually surprising, there is no a priori reason for why it should be true. Taking K-classes $E \mapsto [E]$ is a surjective operation, in the sense that any class $\alpha \in K(X)$ is of the form $[E]$ for a complex $E \in \mathtt{D}(X)$. However, the chern character $\mathrm{ch}\colon K(X) \to H^*(X, \mathbf{Q})$ is very far from being surjective! Indeed, take X an elliptic curve, topologically it's $S^1 \times S^1$ therefore it has non-zero classes in H^1. But ch(E) is even for any E.

Proof. — The proof is actually simple. The inverse of Φ_P is given by a Φ_Q (but going in the opposite direction). We know $\Phi_Q \circ \Phi_P = \Phi_{\mathcal{O}_\Delta}$. I.e. the convolution $Q * P = \mathcal{O}_\Delta$. Passing to cohomology, the convolution $v(Q) * v(P) = v(\mathcal{O}_\Delta)$. So, as silly as it is, if we show that $\Phi^H_{v(\mathcal{O}_\Delta)}$ is the identity, we are actually done. Consider the diagonal map $i\colon X \to X \times X$. We have $\mathcal{O}_\Delta = i_*\mathcal{O}_X$. GRR tells us that

$$\mathrm{ch}(\mathcal{O}_\Delta)\,\mathrm{td}(X \times X) = i_*(\mathrm{ch}(\mathcal{O}_X)\,\mathrm{td}(X)) = i_*\,\mathrm{td}(X)$$

where we used $\mathrm{ch}(\mathcal{O}_X) = 1$. We also have

$$i^*\sqrt{\mathrm{td}(X \times X)} = \mathrm{td}(X)$$

as $T_{X\times X} = T_X \boxplus T_X$.

$$\begin{aligned}
\alpha &\stackrel{?}{=} q_*(v(\mathcal{O}_\Delta)p^*(\alpha)) \\
&= q_*\left(\mathrm{ch}(\mathcal{O}_\Delta)\sqrt{\mathrm{td}(X \times X)}p^*\alpha\right) \\
&= q_*\left(i_*(\mathrm{td}(X))\,\mathrm{td}(X \times X)^{-1}\sqrt{\mathrm{td}(X \times X)}p^*\alpha\right) \\
&= q_*\left(i_*\left(\mathrm{td}(X)i^*\left(\mathrm{td}(X \times X)^{-1}\sqrt{\mathrm{td}(X \times X)}\right)\right)p^*\alpha\right) \\
&= q_*(i_*(1)p^*\alpha) \\
&= q_* i_*(i^*p^*\alpha) \\
&= \alpha.
\end{aligned}$$

□

Remark 15.29. — Note that from the proof we've learned that

$$\mathrm{ch}(\mathcal{O}_\Delta)\sqrt{\mathrm{td}(X \times X)} = i_*(1)$$

Example 15.30. — Let $T\colon \mathtt{D}(X) \to \mathtt{D}(X)$ be $T(E) = E[1]$ the shift by one functor. Then the induced map on $H^*(X)$ is multiplication by -1.

Example 15.31. — Consider the functor $\otimes L$ for L a line bundle. Then the induced map on $H^*(X)$ is multiplication by $\exp(c_1(L))$.

15.4. Hodge theory. — When considering cohomology with complex coefficients we find the Hodge decomposition

$$H^k(X,\mathbf{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$$
$$\overline{H^{p,q}} = H^{q,p}$$
$$H^{p,q}(X) = H^q(X,\Omega^p)$$

Betti numbers are defined as $b_i = \dim H^i(X,\mathbf{Q})$, *hodge* numbers are defined as $h^{p,q} = \dim H^{p,q}(X)$.

It is a fact that chern classes (and all other characteristic classes derived from them) are (p,p) classes. Hence

$$v(?) = \mathrm{ch}(?)\sqrt{\mathrm{td}(X)}\colon K(X) \to \bigoplus_p H^{p,p}(X) \cap H^{2p}(X,\mathbf{Q}).$$

Proposition 15.32. — Let $\Phi_{\mathcal{Q}}\colon \mathtt{D}(X) \to \mathtt{D}(Y)$ be any equivalence. Then $\Phi^H_{\mathcal{Q}}$ induces an isomorphism

$$\bigoplus_{\substack{p-q=i \\ -\dim X \le i \le \dim X}} H^{p,q}(X) \longrightarrow \bigoplus_{\substack{p-q=i \\ -\dim X \le i \le \dim X}} H^{p,q}(Y)$$

Proof. — The key fact to use is that $H^*(X\times Y,\mathbf{Q})$ has a Kunneth decomposition. Concretely, we may write

$$\mathrm{ch}(\mathcal{Q})\sqrt{\mathrm{td}(X\times Y)} = \sum \alpha^{p,q} \boxtimes \beta^{r,s}$$

But since ch and td are (p,p)-classes, only terms with $p+r = q+s$ will be non-zero. Let's have a look at the integral transform. We have

$$\begin{aligned}\Phi(\alpha) &= q_*(p^*\alpha \wedge \sum \alpha^{p,q} \boxtimes \beta^{r,s}) \\ &= q_*(p^*\alpha \wedge p^*(\alpha^{p,q}) \wedge q^*(\beta^{r,s})) \\ &= \sum \int_X (\alpha \wedge \alpha^{p,q})\beta^{r,s}\end{aligned}$$

If $\alpha \in H^{a,b}(X)$, then only the terms where $a+p = \dim X = b+q$ survive. Thus, $a-b = q-p = r-s$. In other words, a class $\alpha \in H^{a,b}(X)$ with $a-b=i$ is sent to a sum of classes $\sum c_{r,s}\beta^{r,s}$ with $r-s=i$. □

15.5. Elliptic curves. — Let C be a curve. If genus of C is different from one, then ω_C is either anti-ample (i.e. $C = \mathbf{P}^1$) or ample. In this case the Bondal–Orlov theorem applies and we have that $D(Y) \simeq D(C)$ if and only if $Y \simeq C$. What happens for elliptic curves? Hodge theory comes to the rescue.

Theorem 15.33. — Let E be an elliptic curve. Then $\mathtt{D}(E) \simeq \mathtt{D}(Y)$ if and only if $E \simeq Y$.

One direction is obvious. Suppose $\mathsf{D}(E) \simeq \mathsf{D}(Y)$, then we know already $1 = \dim E = \dim Y$. We then know that Y must also be an elliptic curve (otherwise we could've applied the Bondal–Orlov theorem). To complete the proof, we need to recall a few basic facts.

Let C be a curve. The *Jacobian* of C is defined as

$$J(C) = \frac{H^0(C, \omega_C)^*}{H_1(C, \mathbf{Z})}$$

where the map $H_1(C, \mathbf{Z}) \to H^0(C, \omega_C)^*$ is defined as

$$\gamma \longmapsto \int_\gamma$$

$$\alpha \mapsto \int_\gamma \alpha$$

so it sends a 1-cycle γ to the functional sending a 1-form α to the integral $\int_\gamma \alpha$.

Here is another thing we can do. Pick a point $p_0 \in C$. We would like to define a map from C to $H^0(C, \omega_C)^*$ by sending p to $\int_{p_0}^{p}(-)$. But for this to be well defined, it should not depend on the choice of a path from p_0 to p. Hence if we mod out by the "periods", i.e. the integrals over closed curves we are in business.

$$C \ni p \longmapsto \int_{p_0}^{p}(-) \in J(C) = \frac{H^0(C, \omega_C)^*}{H_1(C, \mathbf{Z})}$$

This map $C \to J(C)$ is called the *Abel-Jacobi* map. It is a standard fact that E is an elliptic curve if and only if the Abel-Jacobi map is an isomorphism. In other words, $E \simeq \frac{H^0(E,\omega_E)^*}{H_1(E,\mathbf{Z})}$.

The connection to derived equivalences goes via Hodge theory. Recall

$$H^1(C, \mathbf{C}) = H^{1,0}(C) \oplus H^{0,1}(C) = H^0(C, \omega_C) \oplus H^1(C, \mathcal{O}_C)$$

By Serre duality, $H^1(C, \mathcal{O}_C) \simeq H^0(C, \omega_C)^*$. On the other hand, by Poincaré duality $H_1(C, \mathbf{Z}) \simeq H^1(C, \mathbf{Z})$, since these groups are free of finite rank. Our earlier map $H_1(C, \mathbf{Z}) \to H^0(C, \omega_C)^*$ then induces an inclusion

$$H^1(C, \mathbf{Z}) \simeq H_1(C, \mathbf{Z}) \to H^0(C, \omega_C)^* \simeq H^1(C, \mathcal{O}_C) = H^{0,1}(C) \hookrightarrow H^1(C, \mathbf{C})$$

On the other hand, we have the natural inclusion $H^1(C, \mathbf{Z}) \subset H^1(C, \mathbf{Z}) \otimes \mathbf{C} = H^1(C, \mathbf{C})$. One can actually check that these coincide, so that $H^1(C, \mathbf{Z})$ really lands in $H^{0,1}(C)$ in the Hodge decomposition.

Another basic fact is that, for an elliptic curve E, we have $E \simeq H^{0,1}(E)/H^1(C, \mathbf{Z})$. This essentially boils down to the *Abel-Jacobi* map. What is that? Pick a point $p_0 \in C$. We would like to define a map from C to $H^0(C, \omega_C)^*$ by taking p to $\int_{p_0}^{p}$. But, for a 1-form α, the integral $\int_{p_0}^{p} \alpha$ might depend on the path we pick. To remedy this, we mod out by the "periods" i.e. integrals over closed curves. This means we have a well defined map $C \to J(C)$ called the Abel-Jacobi map.

For an elliptic curve, the Abel-Jacobi map is an isomorphism. By invoking Poincaré and Serre as above, we see $E \simeq H^{0,1}(C)/H^1(C, \mathbf{Z})$.

proof of Theorem 15.33. — So, what has the proof boiled down to? We already know Y has to be an elliptic curve. To make sure E and Y are the same elliptic curve, it would suffice to identify $H^1(E, \mathbf{Z})$ with $H^1(Y, \mathbf{Z})$ and $H^{0,1}(E)$ with $H^{0,1}(Y)$, in such a way that the inclusion $H^1(E, \mathbf{Z}) \subset H^{0,1}(E)$ is sent to the corresponding inclusion for Y.

Let $\Phi \colon D(E) \to D(Y)$ be our equivalence, with kernel $Q \in D(E \times Y)$. Since both E and Y are elliptic curves, their tangent bundles are trivial, therefore $td(E \times Y) = 1$. If $\alpha \in H^1(X, \mathbf{Z})$ then $v(Q) \wedge p^*\alpha$ is still an integral class. Indeed, $v(Q) = ch(Q) = r + c_1(Q) +$ stuff. But $r, c_1(Q)$ are integral and the higher order stuff does not contribute to $\Phi^H(\alpha)$. Therefore, we have an isomorphism $H^1(C, \mathbf{Z}) \simeq H^1(Y, \mathbf{Z})$.

Let us have a look at the Hodge theory. We know $\bigoplus_{p-q=i} H^{p,q}$ is preserved. But for $i = -1$ there is only one choice: $H^{0,1}$. So $H^{0,1}(E) \simeq H^{0,1}(Y)$ under Φ.

Finally, since the inclusion $H^1(E, \mathbf{Z}) \subset H^{0,1}(E)$ is basically complexification, the compatibility comes for free. □

Great, we have learned that the derived category of a curve recovers the curve. Turns out that a description of $\mathrm{Aut}(D(C))$ is also possible for elliptic curves, but we might not have time to go into that.

15.6. Mukai pairing. — Recall the Euler pairing χ. If Φ is an equivalence,

$$\chi(E, F) = \chi(\Phi(E), \Phi(F)).$$

The Mukai pairing will be a cohomological shadow of this. Notice that by Riemann-Roch we have

$$\begin{aligned} \chi(E, F) &= \int_X ch(E^\vee)\, ch(F)\, td(X) \\ &= \int_X v(E^\vee) v(F) \end{aligned}$$

It would be nice to write $v(E^\vee)$ purely in terms of $v(E)$. Let us define the *dual* of an even class $e = \sum_i e_i$ with $e_i \in H^{2i}(X, \mathbf{Q})$ as

$$e^\vee = \sum_i (-1)^i e_i$$

We have

$$ch(E^\vee) = ch(E)^\vee.$$

Proposition 15.34. — We have

$$v(E^\vee) = v(E)^\vee \exp(c_1(X)/2).$$

Proof. — Proof boils down to proving that $td(X) = td(X)^\vee \exp(c_1(X))$ which can be shown using the splitting principle, by writing

$$td(X) = \prod \frac{\gamma_i}{1 - \exp(-\gamma_i)}$$

and applying duals. □

Thus we have

$$\chi(E,F) = \int_X \exp(c_1(X)/2) v(E)^\vee v(F).$$

A general cohomology class $\alpha \in H^*(X, \mathbf{C})$ can be written as $\alpha = \sum_j \alpha_j$ with $\alpha_j \in H^j(X, \mathbf{C})$. We define the *dual* as

$$\alpha^\vee = \sum_j \sqrt{-1}^j \alpha_j$$

When α is even this clearly coincides with the dual defined earlier.

The ***Mukai pairing*** of two cohomology classes α, β is defined to be

$$\langle \alpha, \beta \rangle = \int_X \exp(c_1(X)/2) \alpha^\vee \beta.$$

Using this pairing, RR takes the nice compact form

$$\chi(E,F) = \langle v(E), v(F) \rangle$$

Remark 15.35. — If $c_1(X) = 0$, then $\langle , \rangle$ is symmetric if $\dim X$ is even and skew-symmetric if $\dim X$ is odd.

Remark 15.36. — Dualizing is multiplicative: $v^\vee w^\vee = (vw)^\vee$

Remark 15.37. — Let $q\colon X \times Y \to Y$ be the projection, then $q_*(v)^\vee = (-1)^{\dim X} q_*(v^\vee)$ for any $v \in H^*(X \times Y, \mathbf{C})$.

The following fact was shown by Mukai for surfaces and later generalized by Caldararu.

Proposition 15.38. — Let $\Phi\colon \mathtt{D}(X) \to \mathtt{D}(Y)$ be an equivalence. Then

$$\Phi^H\colon H^*(X, \mathbf{Q}) \to H^*(X, \mathbf{Q})$$

is an *isometry* with respect to the Mukai pairing.

Proof. — One needs to show that $\langle v, w \rangle = \langle \Phi(v), \Phi(w) \rangle$. But since Φ is an equivalence, this is the same as showing that

$$\langle \Phi(v), w \rangle = \langle v, \Phi^{-1} w \rangle$$

Let Q be the kernel realizing the equivalence Φ and let $e = v(Q)$ be its Mukai vector. We know a formula for $\Phi^H(\alpha)$, namely $q_*(ep^*\alpha)$. But we also know that Φ^{-1} is given by Fourier-Mukai going in the opposite direction with kernel $Q^\vee \otimes q^*\omega_Y[n]$ with $n = \dim X = \dim Y$. The claim reduces then to a computation using the projection formula and the fact that $[n]$ acts as $(-1)^n$ in cohomology. □

16. Chapter 10: K3 surfaces

16.1. K3 surfaces. — Let's collect some facts about K3 surfaces. A *K3 surface* is (for us) a smooth projective surface X with $\omega_X = \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X)$.

Example 16.1. — A quartic hypersurface $X \subset \mathbf{P}^3$ is a K3 surface (for example you can take $x_0^4 + x_1^4 + x_2^4 + x_3^4$). This follows from adjunction. Indeed, $\omega_X = i^*(\omega_{\mathbf{P}^3} \otimes \mathcal{O}(4))$ where $i\colon X \hookrightarrow \mathbf{P}^3$. Thus, $\omega_X = i^*\mathcal{O}(-4) \otimes \mathcal{O}(4) = \mathcal{O}_X$. On $\mathbf{P}^3$, we have a short exact sequence

$$0 \to \mathcal{O}(-4) \to \mathcal{O}_{\mathbf{P}^3} \to \mathcal{O}_X \to 0$$

applying sheaf cohomology we get

$$H^1(\mathbf{P}^3, \mathcal{O}) \to H^1(\mathbf{P}^3, \mathcal{O}_X) = H^1(X, \mathcal{O}_X) \to H^2(\mathbf{P}^3, \mathcal{O}(-4))$$

where $H^1(\mathbf{P}^3, \mathcal{O}_X) = H^1(X, \mathcal{O}_X)$ follows from the fact that i is a closed embedding (and therefore an affine morphism). But we know (for example by looking it up in Hartshorne) that if $H^i(\mathbf{P}^n, \mathcal{O}(k))$ is non-zero then $i = 0$ or $i = n$. Thus $h_X^{0,1} = 0$.

But there are other ways of constructing K3 surfaces (for example as double covers of $\mathbf{P}^2$ branched over a sextic or Kummer surfaces).

A special case of HRR is *Noether's formula*, which says

$$\chi(X, \mathcal{O}_X) = \frac{c_1^2(X) + c_2(X)}{12}$$

We have $\chi(\mathcal{O}_X) = 2$, $c_1(X) = c_1(T_X) = c_1(\omega_X^\vee) = -c_1(\omega_X) = -c_1(\mathcal{O}_X) = 0$. Thus $24 = c_2(X)$. But $c_2(X) = e(X)$ is the topological Euler characteristic of X, i.e. the alternating sum of its Betti numbers. Let's write the Hodge diamond of X.

$$\begin{array}{ccccc} & & h^{2,2} & & \\ & h^{2,1} & & h^{1,2} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{1,0} & & h^{0,1} & \\ & & h^{0,0} & & \end{array}$$

where the sum of the i-th row is the i-th Betti number $b_i = \dim_{\mathbf{C}} H^i(X, \mathbf{C})$.

Recall that *Hodge symmetry* implies $h^{p,q} = h^{q,p}$ and *Serre duality* implies $h^{p,q} = h^{n-p,n-q}$ where n is the dimension. Indeed, $\overline{H^p(X, \Omega_X^q)} = H^q(X, \Omega_X^p)$. While, $H^{n-q}(X, \Omega_X^{n-p})^\vee = H^q(X, \wedge^{n-p} T_X \otimes \omega_X)$ and $\wedge^{n-p} T_X \otimes \omega_X = \Omega_X^p$.

For our K3, we have $b_1 = h^{0,1} + h^{1,0} = 2h^{0,1} = 0$ and by Poincaré duality $b_3 = b_1 = 0$. We also have $1 = b_0 = b_4$. Since $b_0 - b_1 + b_2 - b_3 + b_4 = e(X) = 24$, it follows that the Hodge diamond is always of the form

$$\begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix}$$

From general facts about Fourier-Mukai transforms, we know that the derived category remembers the sum of the columns in the Hodge decomposition.

16.2. Global Torelli. — All K3 surfaces are actually diffeomorphic (this is why topologists speak of *the* K3 surface) and simply connected. It follows that $H_1(X, \mathbf{Z}) = 0$ and therefore (by universal coefficients) $H^2(X, \mathbf{Z})$ is torsion-free. Cup product (aka intersection) is then a pairing

$$H^2(X, \mathbf{Z}) \times H^2(X, \mathbf{Z}) \to \mathbf{Z}$$

which can be shown to be *even*, in the sense that $\alpha^2 = (\alpha, \alpha) \in 2\mathbf{Z}$ for any $\alpha \in H^2(X, \mathbf{Z})$.

Algebraically, we should think of the map $\mathrm{Pic}(X) \to H^2(X, \mathbf{Z})$ given by taking a line bundle L to its first chern class $c_1(L)$. More geometrically, we are sending a divisor D to the Poincaré dual of the homology class [D]. By the exponential sequence and the fact that $H^1(X, \mathcal{O}_X) = 0$ we have that $H^1(X, \mathcal{O}_X^\times) = \mathrm{Pic}(X) \subset H^2(X, \mathbf{Z})$ is an inclusion (and indeed the map coming from the exponential sequence is the first chern class).

Let ρ be the rank of Pic(X). Since c_1 sends Pic to $H^{1,1}(X)$ we must always have $0 \leq \rho \leq 20$. Notice that $\rho = 0$ means any line bundle is isomorphic to $\mathcal{O}_X$. However, since we assume X to be projective (as opposed to a general complex K3 surface) we actually have $1 \leq \rho \leq 20$. This is because projectivity guarantees an embedding $X \subset \mathbf{P}^N$ for some N. A generic linear subspace $H \subset \mathbf{P}^N$ of the correct codimension will intersect $H \cap X$ in a curve, producing a non-zero divisor class on X (the so-called *hyperplane class*).

So, we have our lattice $H^2(X, \mathbf{Z})$ and its complexification comes with the Hodge decomposition

$$H^2(X, \mathbf{C}) = H^{0,2}(X) \oplus H^{1,1}(X) \oplus H^{2,0}(X)$$

As for elliptic curves, the Hodge structure determines completely the surface.

***Theorem 16.2* (Global Torelli for K3 surfaces). —** Two K3 surfaces X_1, X_2 are isomorphic if and only if the lattices $H^2(X_i, \mathbf{Z})$ are Hodge isometric.

A *Hodge isometry* is an isomorphism of groups $\phi\colon H^2(X_1,\mathbf{Z}) \to H^2(X_2,\mathbf{Z})$ which respects the pairing, and which induces an isomorphism $\phi_{\mathbf{C}}(H^{2,0}(X_1)) = H^{2,0}(X_2)$.

The terminology *period* is sometimes used for the Hodge structure (the period is the one-dimensional subspace $H^{2,0}(X) \subset H^2(X,\mathbf{C})$, i.e. an element of $\mathbf{P}H^2(X,\mathbf{C})$).

16.3. Derived Torelli. — First, a simple fact.

Proposition 16.3. — Assume $D(X) = D(Y)$. If X is K3 then Y is also K3.

Proof. — We know Y is a surface. We also know the order of ω_Y is the same as the order of ω_X, therefore $\omega_Y = \mathcal{O}_Y$. We know that $h_X^{0,1} + h_X^{1,2} = h_Y^{0,1} + h_Y^{1,2}$. Now, $h^{1,2} = h^{2,1} = h^{0,1}$. Therefore $0 = h_X^{0,1} = h_Y^{0,1} = H^1(Y,\mathcal{O}_Y)$. □

The question we want to address is: when are two K3 surfaces derived equivalent? The answer is in terms of (a variant of) the Mukai pairing.

A class in $H^{2*}(X,\mathbf{Z})$ is of the form $(\alpha_0,\alpha_1,\alpha_2)$ with $\alpha_i \in H^{2i}(X,\mathbf{Z})$. We define

$$\langle(\alpha_0,\alpha_1,\alpha_2),(\beta_0,\beta_1,\beta_2)\rangle = \alpha_1\beta_1 - \alpha_0\beta_2 - \alpha_2\beta_0 \in \mathbf{Z}$$

which is minus the Mukai pairing from the previous section. In other words it's the intersection pairing minus the pairing between H^0 and H^4. Notice that in general the Mukai pairing will be $\mathbf{Q}$-valued, but c_1 of a K3 surface is zero so no fractions are introduced.

By relabelling, we define $\tilde{H}^*(X,\mathbf{Z})$ to be $H^{2*}(X,\mathbf{Z})$, the even cohomology of X, equipped with the following Hodge structure.

$$\tilde{H}^{2,0}(X) = H^{2,0}(X) \qquad \tilde{H}^{1,1}(X) = (H^0 \oplus H^4)(X) \oplus H^{1,1}(X)$$

The "twist" in the grading of the Hodge structure is for (derived) convenience. After all, derived categories remember the vertical lines in the Hodge diamond, so $\tilde{H}$ is precisely collapsing the middle vertical line to a single piece of weight two.

In any case, with this convention the Mukai vector $v(E) \in H^*(X,\mathbf{Z})$ of a complex lives in $\tilde{H}^{1,1}(X)$. Notice that $v(E) \in H^*(X,\mathbf{Z})$ because X is a K3 surface (in general you would have fractions). Indeed, $\mathrm{td}(X) = 1 + c_1/2 + (c_1^2 + c_2)/12 + \cdots$. The higher order terms all vanish as X is a surface. Moreover, $c_1(X) = 0$ so $\mathrm{td}(X) = 1 + \mathrm{td}_2(X) = 1 + c_2/12$. But we know that $c_2(X) = e(X) = 24$ so $\mathrm{td}(X) = (1,0,2)$.

***Lemma 16.4* (Mukai).** — Let $P \in D(X_1 \times X_2)$ with X_i a K3 surface. Then $v(P) \in H^*(X \times Y,\mathbf{Z})$.

We omit the proof, which is a computation of characteristic classes plus GRR.

***Corollary 16.5* (Mukai).** — If $\Phi\colon D(X_1) \to D(X_2)$ is an equivalence, then $\Phi^H\colon \tilde{H}^*(X_1,\mathbf{Z}) \to \tilde{H}^*(X_2,\mathbf{Z})$ is a Hodge isometry.

Proof. — We already know that $\Phi^H\colon H^*(X_1,\mathbf{Q}) \to H^*(X_2,\mathbf{Q})$ is an isometry with respect to the Mukai pairing. By the the proposition above, it restricts to an isometry with integer coefficients. Since we know the sum of the Hodge pieces with $p - q = i$ are preserved, we conclude. □

***Theorem 16.6* (Mukai-Orlov).** — Two K3 surfaces X_1, X_2 are derived equivalent if and only if there is a Hodge isometry $\tilde{H}^*(X_1, \mathbf{Z}) \simeq \tilde{H}^*(X_2, \mathbf{Z})$.

Before we prove this, some observations.

One, the shift by one functor [1] acts as multiplication by −1 in cohomology.

Two, if $L \in \mathrm{Pic}(X)$ is a line bundle, then tensoring by it defines an equivalence $\Phi\colon \mathrm{D}(X) \to \mathrm{D}(X)$. Earlier we checked that its effect on cohomology is multiplication by $\mathrm{ch}(L) = \exp(c_1(L)) = 1 + c_1(L)$, which is an integral class. Thus, for $\alpha = (r, l, s)$

$$\begin{aligned}\Phi^H(\alpha) &= \alpha.\,\mathrm{ch}(L)\\ &= (r, l, s) + (0, rc_1(L), l.c_1(L))\\ &= (r, l + rc_1(L), s + l.c_1(L))\end{aligned}$$

Three, the structure sheaf $\mathcal{O}_X$ is what is called a *spherical object.* Attached to any such object there is a *spherical twist* $T_{\mathcal{O}_X}\colon \mathrm{D}(X) \to \mathrm{D}(X)$ which is an autoequivalence (this was defined by Seidel and Thomas, see details below). One checks that

$$T^H_{\mathcal{O}_X}(\alpha) = \alpha + \langle \alpha, (1, 0, 1)\rangle (1, 0, 1).$$

These three operations will be useful in the proof below to manipulate cohomology classes.

Proof of Theorem above. — One direction we already know. Suppose we have a Hodge isometry ϕ. We will see that, similarly to elliptic curves, the whole problem is governed by a single vector. In this case, $\phi(0, 0, 1) = v = (r, l, s)$, the image of the Mukai vector of a point.

Suppose $v = (0, 0, 1)$. Then ϕ restricts to a Hodge isometry $H^*(X_1, \mathbf{Z}) \simeq H^2(X_2, \mathbf{Z})$. Therefore, by classic Torelli, X_1 and X_2 are isomorphic.

Suppose now $r \neq 0$. By applying the equivalence [1], we may assume $r > 0$. We now appeal to a general theorem on moduli of sheaves.

Claim 16.7. — Suppose Y is a K3 surface and $v, v' \in \tilde{H}^{1,1}(X, \mathbf{Z})$ are two classes with $\langle v, v\rangle = 0$ and $\langle v, v'\rangle = 1$. Then there exists another K3 surface M and a sheaf $E \in \mathrm{D}(Y \times M)$ such that the fibre $v(E_m) = v$ for all fibres E_m of E at $m \in M$ and such that the transform $\Phi_E\colon \mathrm{D}(Y) \to \mathrm{D}(M)$ is an equivalence.

In our setup, $\langle v, v\rangle = (0, 0, 1)^2 = 0$. Take $v' = \phi(-1, 0, 0)$, then $\langle v, v'\rangle = 1$. Thus there exists a K3 surface M satisfying the assumptions as above. In particular, the composition

$$\psi\colon \tilde{H}^*(X_1, \mathbf{Z}) \xrightarrow{\phi} \tilde{H}^*(X_2, \mathbf{Z}) \xrightarrow{\Phi^H_E} \tilde{H}^*(M, \mathbf{Z})$$

is a Hodge isometry sending $\psi(0, 0, 1) = (0, 0, 1)$. By classic Torelli, $X_1 \simeq M$. Therefore, the isomorphism $X_1 \simeq M$ composed with the inverse of the equivalence Φ_E yields an equivalence $\mathrm{D}(X_1) \simeq \mathrm{D}(X_2)$.

Finally, suppose now $v = (0, l, s)$ with $l \neq 0$. If $s \neq 0$ we leave v alone. If not, we tensor by a line bundle, for the following reason. Recall that, if $L \in \mathrm{Pic}(X)$, tensoring with it has the effect of sending $(0, l, 0)$ to $(0, l, l.c_1(L))$. We then find L such that $l.c_1(L) \neq 0$. Now we apply the spherical twist around $\mathcal{O}_{X_2}$ (see below for more details). By composing the two operations above, we see that v may be sent to a class of the form (r', l', s') with $r' \neq 0$. Hence we reduce to the first case of the proof. □

It goes without saying that the proof we just gave is highly unsatisfactory. Not only have we treated classic Torelli as a mysterious black box, but the true heavy lifting was performed by Claim 16.7 which we have not explained.

16.4. Spherical Twists. — Let $S \in \mathrm{D}(X)$. If E is any other object, we may consider the evaluation map

$$\mathrm{RHom}(S, E) \otimes_{\mathbf{C}} S \to E \tag{16.1}$$

and we call $T_S(E)$ its cone.

Assume now X is a K3 surface. We say S is *spherical* $\mathrm{Hom}(S, S) = \mathbf{C} = \mathrm{Hom}(S, S[2])$ and $\mathrm{Hom}(S, S[i]) = 0$ for all other i. In that case, Seidel and Thomas showed the functor $T_S \colon \mathrm{D}(X) \to \mathrm{D}(X)$ is an equivalence.

Remark 16.8. — Spherical twists provide the first interesting auto-equivalence of $\mathrm{D}(X)$ which is not visible at the level of $\mathrm{Coh}(X)$. Their definition was actually inspired by mirror symmetry (spherical twists are mirror to Dehn twists).

Remark 16.9. — Taking cones is not functorial, so defining T as a cone is problematic. Two solutions: use dg-categories or define T in terms of kernels.

Any line bundle L on a K3 surface is spherical. In particular, $\mathcal{O}_X$ is spherical. Let's define $T_{\mathcal{O}_X}$ in terms of kernels. We notice that $E \mapsto E$ is the identity, so the kernel is $\mathcal{O}_\Delta \in \mathrm{D}(X \times X)$. While $E \mapsto H^*(X, E) \otimes_{\mathbf{C}} \mathcal{O}_X$ has kernel $\mathcal{O}_{X\times X}$. Indeed, $q_*(\mathcal{O}_{X\times X} \otimes p^*E) = q_*p^*E = H^*(X, E) \otimes_{\mathbf{C}} \mathcal{O}_X$. There is a natural map $\mathcal{O}_{X\times X} \to \mathcal{O}_\Delta$, which corresponds to the evaluation map in (16.1). Its cone is precisely I[1], the ideal sheaf of the diagonal shifted by one. Thus, $T_{\mathcal{O}_X}(E) = \Phi_{I[1]}(E)$. But what we care about is the effect on cohomology. If α is a cohomology class, then $T^H_{\mathcal{O}_X}(\alpha) = \Phi^H_{\mathcal{O}_\Delta}(\alpha) - \Phi_{\mathcal{O}^H_{X\times X}}(\alpha)$. Notice that $\mathrm{td}(X) = (1, 0, 2)$ so that $\sqrt{\mathrm{td}(X)} = (1, 0, 1)$. Moreover, $\sqrt{\mathrm{td}(X \times X)} = p^*(1, 0, 1).q^*(1, 0, 1)$. Thus,

$$\begin{aligned}\Phi^H_{\mathcal{O}_{X\times X}}(\alpha) &= q_*(p^*(1, 0, 1).q^*(1, 0, 1).p^*\alpha)\\ &= q_*p^*(\alpha.(1, 0, 1))(1, 0, 1)\\ &= -\langle \alpha, (1, 0, 1)\rangle(1, 0, 1).\end{aligned}$$

Assembling things together,

$$T^H_{\mathcal{O}_X}(\alpha) = \alpha + \langle \alpha, (1, 0, 1)\rangle(1, 0, 1)$$

which can be thought as reflection about the vector $(1, 0, 1)$.

Thus we have $T^H_{\mathcal{O}_X}(0, l, s) = (0, l, s) - s(1, 0, 1) = (-s, l, 0)$. Which is what we needed to finish off the proof above.

17. Outroduction

As is common practice in math texts, we now make the standard (and ridiculous) assumption that a reader exists and is still paying attention. Yet more boldly, we assume such superhuman reader craves for yet more derived enlightenment. To her or him, we have the following suggestions.

Concerning Huybrechts's book, there have been a few grave omissions: Orlov's blow up formula, Beilinson's theorem for projective space, Mukai's original result for abelian varieties, the Bondal-Orlov-Bridgeland criterion for equivalences and the entirety of Chapter 11 (flips and flops).

Going beyond Huybrechts's book, two topics have dominated the field in recent years and deserve more attention: stability conditions and semi-orthogonal decompositions. For a treatment of the former, there are excellent notes by Huybrechts (no surprises here) and a forthcoming book by Bayer-Macrì. For semi-orthogonal decompositions (and the beautiful framework of homological projective duality) there are a few sources. We mention Galkin's and Logvinenko's notes (which can be found on their respective websites) and Kuznetsov's survey on rationality questions.

Happy deriving.

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