Test 1: Math 212

Practice Exam

This is a practice exam for Math 212 Exam I, Spring 2008. It is LONGER than the in class exam will be, but should give you a good idea for the standard types of problems and difficulty level that will be on the exam. Problems like these will constitute roughly 80% of the exam. The rest of the exam will consist of problems that you have not seen before. Let me know if you see any typos or incorrect solutions!!

Problem 1

Find an equation of the plane in $\mathbb{R}^3$ which is tangent to the surface

$$x^3 - 2y^3 + xz^2 = 0$$

at the point $(1, 1, 1)$.

The surface is the level surface given by $g(x, y, z) = x^3 - 2y^3 + xz^2 = 0$. The gradient is $\nabla g = (3x^2 + z^2, -6y, 2xz)$. Therefore, $\nabla g(1, 1, 1) = (4, -6, 2)$ is the normal vector to the surface. The equation of the plane is given by

$$(4, -6, 2).(x - 1, y - 1, z - 1) = 4x - 6y + 2z = 0.$$

Problem 2 Sketch the level curves for the function $f(x, y) = 2 - x^2 - y^2$ for $c = -1, 0, 1$.

Solution: Concentric circles of radius $\sqrt{3}, \sqrt{2}, 1$ all centered at the origin.

Problem 3 Describe the sections $x = 1$ and $y = x$ for the graph of $f(x, y) = x^2 - 3y^2$.

Solution: The section $x = 1$ is the set of points $(1, y, 1 - 3y^2)$. This consists of the parabola $(y, 1 - 3y^2)$ in the plane $x = 1$. The section $y = x$ is the set of points $(x, x, -2x^2)$ which is also a parabola in the $y = x$ plane.

Problem 4

- What is the *del operator* ∇? (Write an equation $\nabla\ = ???$)

  Solution:
  $$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$$

- Let
  $$\overline{F}(x,y,z) \ = \ 2xye^z\,\hat{i} + e^z x^2\,\hat{j} + (x^2ye^z + z^2)\,\hat{k}.$$

  Compute the following:

  a) $\operatorname{div}\overline{F} = \nabla\cdot\overline{F}$

  Solution:
  $$2ye^z + 0 + x^2ye^z + 2z = 2z + e^z(2y + x^2y).$$

  b) $\operatorname{curl}\overline{F} = \nabla\times\overline{F}$

  Solution:
  $$\operatorname{curl}\overline{F} = 0.$$

  c) $\operatorname{div}\overline{F}\,(1,-2,0)$

  Solution:
  $$-6.$$

  d) Is $\overline{F}$ expanding or contracting?

  Solution: Contracting.

Problem 5

- Calculate the derivative $\mathbf{D}f(x,y)$ for $f(x,y) = \left(x^2y,\, e^{-xy}\right)$

  Solution:
  $$\begin{pmatrix} 2xy & x^2 \\ -ye^{-xy} & -xe^{-xy} \end{pmatrix}$$

- Let $g(x,y) = xy+3x$ and suppose that $\bar{c}(t) = (x(t), y(t))$ is a differentiable path in $\mathbb{R}^2$. Let $h(t) = g \circ c\,(t)$.

  Compute $\mathbf{D}h(t)$ in terms of $x(t)$, $y(t)$, $\frac{dx}{dt}$, and $\frac{dy}{dt}$.

  Solution:
  $$\mathbf{D}h(t) = (y(t)+3)\frac{dx}{dt} + x(t)\frac{dy}{dt}.$$

Problem 6

Suppose that the temperature in a room is governed by the function

$$T(x, y, z) = -2x + y^2 \sin(x^2) + 2z + 78.$$

- What is the unit vector which indicates the direction to move from $(0, 0, 0)$ so that the temperature **decreases** as fast as possible?

  Solution: The gradient vector at $(0, 0, 0)$ is $(-2, 0, 2)$. Therefore to move in the direction of greatest **decrease**, from the origin, you would move in the direction $-(-2, 0, 2) = (2, 0, -2)$. The unit vector in this direction is $\frac{\sqrt{2}}{2}(1, 0, -1)$

- From the origin, what is the rate of change of the temperature in direction

$$\bar{v} = \frac{\sqrt{3}}{3}\left(-\hat{\imath} - \hat{\jmath} + \hat{k}\right)?$$

  Solution: Compute the directional derivative: $\nabla T(0, 0, 0).\bar{v}$:

$$(-2, 0, 2).\ \frac{\sqrt{3}}{3}(-1, -1, 1) = \frac{4\sqrt{3}}{3}$$

Problem 7

Consider the path

$$\bar{c} : [0, 1] \longrightarrow \mathbb{R}^3$$

given by

$$\bar{c}(t) = \left(2t, \frac{4t^{3/2}}{3}, \frac{t^2}{2}\right).$$

- Compute the length of the path.

  Solution: By the arc length formula, the Arc length of $\bar{c}$ is

  $$\int_0^1 ||\bar{c}'(t)||\, dt = \int_0^1 t + 2\, dt = 5/2$$

- What is the velocity of $\bar{c}$ at $t = 1/4$.

  Solution: $\bar{c}'(1/4) = (2, 1, \frac{1}{4})$.

Problem 8

Let L_1 and L_2 be the intersecting lines given by the paths

$$\begin{aligned} L_1(t) &= (3,6,1) + t(1,2,1) \\ L_2(t) &= (2,1,-3) + t(-1,1,2). \end{aligned}$$

- What is their point of intersection?

  Solution: $(1,2,-1)$.

- Find the angle between L_1 and L_2.

  Solution: $\pi/3$

- Find a vector that is perpendicular to both L_1 and L_2.

  Solution: $(3,-3,3)$

Problem 9 Let D be the region bounded by the positive x and y-axes and the line $3x + 4y = 10$. Compute

$$\iint_D (x^2 + y^2)\, dA$$

Solution:

$$\frac{5}{24}\left(\frac{10}{3}\right)^3 + \frac{1}{9}\left(\frac{5}{2}\right)^4$$

Problem 10 Does there exist a function f such that the gradient of f is

$$(x + y^2)\mathbf{i} + (x - y^2)\mathbf{j}?$$

If so, find such an f; if not, argue convincingly why no such f can exist.

Solution:Calculate

$$\mathrm{curl}((x + y^2)\mathbf{i} + (x - y^2)\mathbf{j}) = (1 - 2y)\mathbf{k}$$

Since the curl is not identically zero, there is no such function f.

Problem 11 Compute

a.

$$\int_0^1 \int_{x^2}^1 \sin(y^{3/2}) dy\, dx.$$

Solution: $\frac{4}{3} \sin\left(\frac{1}{2}\right)^2$

b. Let R denote the region bounded by $y = x$ and $y = x^2$. Compute

$$\int\int_R xy\, dA$$

Solution: $\frac{1}{24}$

c.

$$\int_0^1 \int_{\sqrt{y}}^1 \cos(x^3) dx\, dy.$$

Solution: $\frac{\sin(1)}{3}$.

d.

$$\int_1^2 \int_1^4 xy + \frac{x}{y+1}\, dx\, dy$$

Solution: $11\frac{1}{4} + \frac{15}{2} \ln\left(\frac{3}{2}\right)$.

Problem 12 Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function. State what it means for f to be differentiable at a point $\bar{x}_0 \in \mathbb{R}^n$.

Problem 13 Calculate the following limits if they exist:

a.

$$\lim_{(x,y)\to(0,0)} \frac{e^{xy} - 1}{y}$$

Solution: 0

b.

$$\lim_{(x,y)\to(0,0)} \frac{y^2}{x^2 + y^2}$$

Solution: DNE