

Math 212 *Multivariable Calculus* - Final Exam

Instructions: You have **3 hours** to complete the exam (**12 problems**). This is a closed book, closed notes exam. Use of calculators is not permitted. Show all your work for full credit. **Please do not forget to write your name and your instructor's name on the blue book cover, too.**

Print your instructor's name : _____

Print your name : _____

Upon finishing please sign the pledge below:

On my honor I have neither given nor received any aid on this exam.

Signature : _____

Problem	Max Points	Your Score	Problem	Max Points	Your Score
1	8		7	8	
2	8		8	8	
3	8		9	9	
4	8		10	9	
5	8		11	9	
6	8		12	9	
			Total	100	

[1] (8 points) Find the equation of the tangent line to the path $\mathbf{c}(t) = (\cos t, \sin t, t^2)$ at $t = \pi$.

Solution. The derivative of this path is given by

$$\mathbf{c}'(t) = (-\sin(t), \cos(t), 2t)$$

The tangent line to $c(t)$ at the point $t = \pi$ is given by

$$\begin{aligned} L(t) &= c(\pi) + t \cdot c'(\pi) \\ &= (\cos(\pi), \sin(\pi), \pi^2) + t \cdot (-\sin(\pi), \cos(\pi), 2\pi) \\ &= (-1, 0, \pi^2) + t \cdot (0, -1, 2\pi) \\ &= (-1, -t, \pi^2 + 2\pi t) \end{aligned}$$

(Or you may use $L(t) = c(\pi) + (t - \pi) \cdot c'(\pi)$.)

[2] (8 points) Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ be defined by:

$$f(x, y, z) = 3z + e^{x^2 - y^2}$$

Let C be the set of the heads of unit vectors v in \mathbb{R}^3 such that f increases at $1/3$ of its maximum rate of change in the direction v starting from $(0, 0, 1)$. Find the equation(s) which determine(s) the set C . (Hint : C is a circle in \mathbb{R}^3 .)

Solution. The gradient of f is given by

$$\nabla f = (2x \exp(x^2 - y^2), -2y \exp(x^2 - y^2), 3)$$

$\nabla f(0, 0, 1)$ is a vector in the direction of greatest increase for the function f at the point $(0, 0, 1)$. So

$$\begin{aligned} \nabla f(0, 0, 1) &= (0, 0, 3) \\ &= 3 \cdot (0, 0, 1) \end{aligned}$$

Calculate the maximum rate of change of f by evaluating the directional derivative of f .

$$\begin{aligned} Df_{(0,0,1)} &= \nabla f \cdot (0, 0, 1) \\ &= (0, 0, 3) \cdot (0, 0, 1) \\ &= 3 \end{aligned}$$

$1/3$ of this rate is just 1, so the goal of this problem is to find an equation for the unit vectors $v = (x, y, z)$ such that $Df_v = 1$ at the point $(0, 0, 1)$.

$$\begin{aligned} 1 &= Df_v \\ &= \nabla f \cdot v \\ &= (0, 0, 3) \cdot (x, y, z) \\ &= 3z \\ 1/3 &= z \end{aligned}$$

Now applying the constraint that v be a unit vector

$$\begin{aligned} v \cdot v &= 1 \\ (x, y, 1/3) \cdot (x, y, 1/3) &= 1 \\ x^2 + y^2 + 1/9 &= 1 \\ x^2 + y^2 &= 8/9 \end{aligned}$$

The final equations are : $x^2 + y^2 = 8/9$ and $z = 1/3$.

[3] (8 points) Find the absolute minimum and maximum for the function $f(x, y) = x + 2y^2 + 1$ on the unit disk $D = \{(x, y) | x^2 + y^2 \leq 1\}$.

Solution. On the interior of D , which is $\{(x, y) | x^2 + y^2 < 1\}$, if (x, y) is a critical point of f , $\nabla f(x, y) = (0, 0)$. But $\nabla f(x, y) = (1, 4y) \neq (0, 0)$. Hence f has its extrema on ∂D .

On $\partial D = \{(x, y) | x^2 + y^2 = 1\}$, we use Lagrange Multiplier. Let $g(x, y) = x^2 + y^2$, and solve $\nabla f(x, y) = \lambda \nabla g(x, y)$ and $x^2 + y^2 = 1$ simultaneously. Then we have $1 = 2\lambda x$, $4y = 2\lambda y$, and $x^2 + y^2 = 1$.

If $y = 0$, $x = \pm 1$, hence we have $(x, y) = (1, 0), (-1, 0)$. If $y \neq 0$, then $\lambda = 2$ and $x = \frac{1}{4}$, hence $y = \pm \frac{\sqrt{15}}{4}$. Thus $(x, y) = (\frac{1}{4}, \frac{\sqrt{15}}{4}), (\frac{1}{4}, -\frac{\sqrt{15}}{4})$.

Now $f(1, 0) = 2$, $f(-1, 0) = 0$, $f(\frac{1}{4}, \frac{\sqrt{15}}{4}) = f(\frac{1}{4}, -\frac{\sqrt{15}}{4}) = \frac{25}{8}$. Therefore f has the absolute maxima $(\frac{1}{4}, \frac{\sqrt{15}}{4})$ and $(\frac{1}{4}, -\frac{\sqrt{15}}{4})$, and the absolute minimum $(-1, 0)$.

[4] (8 points) Let C be a simple closed curve in \mathbb{R}^3 , and $\mathbf{c} : [a, b] \rightarrow \mathbb{R}^3$ be a parametrization of C such that $\|\mathbf{c}'(t)\| = 1$ for all $t \in [a, b]$. The *curvature* of C at $\mathbf{c}(t)$ is defined to be

$$\kappa(t) = \|\mathbf{c}''(t)\|$$

and the *total curvature* of C is defined by

$$\int_{\mathbf{c}} \kappa \, ds.$$

Compute the total curvature of a circle $C = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 = R^2, z = 0\}$.

Solution. Let $\mathbf{c}(s) = (R\cos(s/R), R\sin(s/R), 0)$, $s \in [0, 2\pi]$. Note that

$$\mathbf{c}'(s) = (-\sin(s/R), \cos(s/R), 0)$$

is of unit length for all s . We have $\mathbf{c}''(s) = (-\frac{\cos(s/R)}{R}, -\frac{\sin(s/R)}{R}, 0)$ and $\kappa(s) = \|\mathbf{c}''(s)\| = 1/R$ for all s . Hence the total curvature is

$$\int_{\mathbf{c}} \kappa \, ds = \int_0^{2\pi} \frac{1}{R} \, ds = \frac{1}{R} \cdot \text{length}(C) = \frac{1}{R} \cdot 2\pi R = 2\pi.$$

[5] (8 points) Let C be the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{s}$ where $\mathbf{F}(x, y) = (x/a^2, y/b^2)$.

Solution. Method I: Consider the region

$$S = \{(x, y) \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1\}.$$

By Stokes' theorem, we have

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}.$$

But $\nabla \times \mathbf{F} = 0$, hence the integral is 0.

Method II: The ellipse can be parametrized by

$$\mathbf{c}(t) = (a \cos t, b \sin t), \quad t \in [0, 2\pi].$$

Since we have

$$\mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) = \frac{\cos t}{a}(-a \sin t) + \frac{\sin t}{b}b \cos t = 0,$$

the line integral is zero.

[6] (8 points) Evaluate the line integral $\int_{\mathbf{c}} e^{x^2} dx - xy dy + y^2 dz$, where $\mathbf{c}(t) = (1, t, t^2)$, $0 \leq t \leq 1$.

Solution. $\int_{\mathbf{c}} e^{x^2} dx - xy dy + y^2 dz = \int_0^1 (e \cdot 0 - 1 \cdot t \cdot 1 + t^2 \cdot 2t) dt = \int_0^1 (-t + 2t^3) dt = \left[-\frac{1}{2}t^2 + \frac{2}{4}t^4 \right]_0^1 = 0.$

[7] (8 points) Find the area of the portion of the unit sphere inside the cylinder $x^2 + y^2 = \frac{1}{2}$ and $z > 0$.

Solution. The intersection of the unit sphere and the cylinder is a circle, and the angle between the z -axis and a line from the origin to the circle is $\frac{\pi}{4}$. Denote the surface by S . We parametrize S by $\Phi(\phi, \theta) = (\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi)$ where $0 \leq \phi \leq \frac{\pi}{4}$, $0 \leq \theta \leq 2\pi$. Then $\|\Phi_\phi \times \Phi_\theta\| = \sin \phi$.

$$\text{Hence } \iint_S dS = \int_0^{2\pi} \int_0^{\pi/4} \sin \phi \, d\phi \, d\theta = [-\cos \phi]_0^{\pi/4} \cdot 2\pi = (2 - \sqrt{2})\pi.$$

[8] (8 points) W is the volume defined by $x^2 + y^2 + z^2 \leq 1$ and $y \leq x$. Find the flux of $(x^3 - 3x, y^3 + xy, z^3 - xz)$ out of W .

Solution. Calculate that

$$\begin{aligned} \nabla \cdot F &= 3x^2 - 3 + 3y^2 + x + 3z^2 - x \\ &= 3(x^2 + y^2 + z^2 - 1) \end{aligned}$$

By Gauss Theorem the flux out of W is given by

$$\begin{aligned}\iint_{\delta W} \mathbf{F} \cdot d\mathbf{S} &= \iiint_W \nabla \cdot \mathbf{F} \cdot dW \\ &= 3 \iiint_W x^2 + y^2 + z^2 - 1 dx dy dz\end{aligned}$$

In spherical coordinates, the defining equations of the volume W are $\rho^2 \leq 1$ and $\sin\theta \leq \cos\theta$. The latter inequality implies $-3\pi/4 \leq \theta \leq \pi/4$.

The Jacobian determinant for spherical coordinates is $\rho^2 \sin\phi$, so the integral is just:

$$\begin{aligned}3 \iiint_W x^2 + y^2 + z^2 - 1 dx dy dz &= 3 \int_0^1 \int_{-3\pi/4}^{\pi/4} \int_0^\pi (\rho^2 - 1) \rho^2 \sin\phi d\phi d\theta d\rho \\ &= -4\pi/5\end{aligned}$$

[9] (9 points) Let S be the surface of the tetrahedron whose vertices are $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 2)$ and the origin. Evaluate $\iint_S f \, dS$ where $f(x, y, z) = xz$.

Solution. Let S' , S'' , S''' , T denote the faces of the tetrahedron in the xz -plane, yz -plane, xy -plane, and the plane $2x + 2y + z = 2$ (the frontal face), respectively. Since $x = 0$ on S'' and $z = 0$ on S''' , it follows that the integral vanishes on those faces. Hence we have

$$(1) \quad \iint_S xz \, dS = \iint_{S'} xz \, dS + \iint_T xz \, dS.$$

$$\begin{aligned}\iint_{S'} xz \, dS &= \int_0^1 \int_0^{1-x} xz \, dz \, dx \\ &= \int_0^1 x(1-x)^2/2 \, dx = 1/24.\end{aligned}$$

(2) T is the graph of $z = 2 - 2x - 2y$ over the triangle Δ with vertices $(1, 0, 0)$, $(0, 1, 0)$, and the origin. Let $g(x, y) = 2 - 2x - 2y$. Then we have

$$\begin{aligned}\iint_T xz \, dS &= \iint_{\Delta} xg(x, y) \sqrt{1 + g_x^2 + g_y^2} \, dA \\ &= \int_0^1 \int_0^{1-x} x(2 - 2x - 2y)3 \, dy \, dx \\ &= \int_0^1 3x(1-x)^2 \, dx = 1/4.\end{aligned}$$

$$(3) \quad \iint_S xz \, dS = 1/24 + 1/4 = 7/24.$$

[10] (9 points) Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where

$$\mathbf{F}(x, y, z) = (1, 1, z(x^2 + y^2)^2)$$

and S is the surface of the cylinder $x^2 + y^2 \leq 1$, $0 \leq z \leq 1$ (oriented with the outward unit normal).

Solution. To use Gauss Theorem, write $\delta W = S$ and so

$$\begin{aligned} \int \int_S \mathbf{F} \cdot d\mathbf{S} &= \int \int_{\delta W} \mathbf{F} \cdot d\mathbf{S} \\ &= \int \int \int_W \operatorname{div}(\mathbf{F}) \cdot dW \end{aligned}$$

$$\begin{aligned} \operatorname{div}(\mathbf{F}) &= 0 + 0 + (x^2 + y^2)^2 \\ &= (x^2 + y^2)^2 \end{aligned}$$

The Jacobian determinant of cylindrical coordinates is r , so

$$\begin{aligned} \int \int \int_W \operatorname{div}(\mathbf{F}) \cdot dW &= \int \int \int_W (x^2 + y^2)^2 \cdot dW \\ &= \int_0^1 \int_0^{2\pi} \int_0^1 (r^2)^2 r dz d\theta dr \\ &= \pi/3 \end{aligned}$$

[11] (9 points) Let S be the portion of the unit sphere centered at the origin that is cut out by the cone $z \geq \sqrt{x^2 + y^2}$. Evaluate $\int \int_S \mathbf{F} \cdot d\mathbf{S}$ where

$$\mathbf{F}(x, y, z) = (xy + \cos z, -yx + x^2 + z^3, 2z^2 + x).$$

Solution. The boundary C of S is the circle of radius $1/\sqrt{2}$ on $z = 1/\sqrt{2}$ and with its center on the z -axis. Let D be the disc in the plane $z = 1/\sqrt{2}$ bounded by C , and $T = S \cup D$. Then we have

$$\int \int_S \mathbf{F} \cdot d\mathbf{S} = \int \int_T \mathbf{F} \cdot d\mathbf{S} - \int \int_D \mathbf{F} \cdot d\mathbf{S}.$$

(1) T is a surface without boundary and we can apply Gauss' theorem to the first integral $\int \int_T \mathbf{F} \cdot d\mathbf{S}$. Let W be the (3-dimensional) region bounded by T .

$$\begin{aligned} \int \int_T \mathbf{F} \cdot d\mathbf{S} &= \iiint_W \nabla \cdot \mathbf{F} \, dV \\ &= \iiint_W y - x + 4z \, dV. \end{aligned}$$

But $\iiint_W y \, dV = \iiint_W x \, dV = 0$ because of the apparent symmetry.

$$\begin{aligned} \iiint_W 4z \, dV &= \int_0^{2\pi} \int_{1/\sqrt{2}}^1 \int_0^{\sqrt{1-z^2}} 4z r \, dr \, dz \, d\theta \\ &= 8\pi \int_{1/\sqrt{2}}^1 z \frac{1-z^2}{2} \, dz = \frac{\pi}{4}. \end{aligned}$$

(2) Since $\mathbf{n} = -\mathbf{k}$ on D , we have

$$\begin{aligned} \int \int_D \mathbf{F} \cdot d\mathbf{S} &= \int \int_D \mathbf{F} \cdot (-\mathbf{k}) \, dS \\ &= - \int \int_D 2z^2 + x \, dA. \end{aligned}$$

But $\iint_D x \, dA = 0$ because of symmetry.

$$\begin{aligned} - \iint_D 2z^2 \, dA &= - \iint_D 2 \cdot \frac{1}{2} \, dA \\ &= - \iint_D dA = -\pi/2. \end{aligned}$$

$$(3) \iint_S \mathbf{F} \cdot d\mathbf{S} = \pi/4 - (-\pi/2) = 3\pi/4.$$

[12] (9 points) Evaluate $\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = (xy, y, z)$ and S is the surface described by $x^2 + y^2 + z^2 = 2$ and $z \geq 1$. (S is oriented with the upward unit normal.)

Solution. ∂S is a circle with the equations $x^2 + y^2 = 1$ and $z = 1$. Use Stokes' Theorem.

$$\begin{aligned} \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} &= \int_{\partial S} \mathbf{F} \cdot d\mathbf{s}. \text{ We parametrize } \partial S \text{ by } \mathbf{c}(t) = (\cos t, \sin t, 1), 0 \leq t \leq 2\pi. \\ \int_{\partial S} \mathbf{F} \cdot d\mathbf{s} &= \int_0^{2\pi} (\cos t \sin t (-\sin t) + \sin t \cos t + 1 \cdot 0) \, dt = \int_0^{2\pi} (-\sin^2 t \cos t + \frac{1}{2} \sin 2t) \, dt = \\ &= \left[-\frac{1}{3} \sin^3 t - \frac{1}{4} \cos 2t \right]_0^{2\pi} = 0. \end{aligned}$$