Practice Problems for Math 212 Final

April 19, 2006

Instructions: Below are many practice problems for the final exam in MATH 212. You should know how to do all of these problems, but this list may not comprise **all** of the material to be tested on the final exam.

Note: Two points will be deducted each time you fail to quote (i.e. indicate you are using) a major theorem. For example, if you are using Green's theorem , you must indicate it: $\int_C P\ dx + Q\ dy =^{GT} \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\ dx\ dy$.

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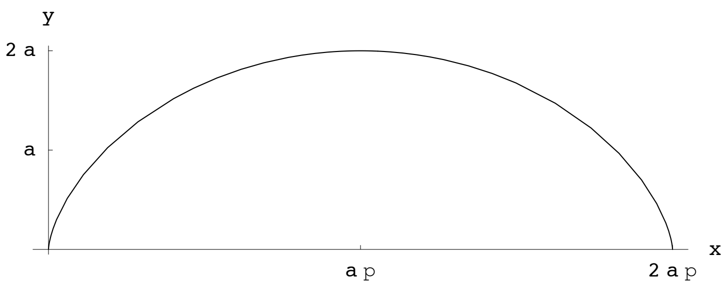
1. Evaluate the integral $\int_C (2x^3 - y^3)\ dx + (x^3 + y^3)\ dy$ where C is the unit circle oriented positively.

2. Find the area bounded by the x-axis and one arc of the cycloid, which is parameterized by

$$\begin{aligned} x &= a(\theta - \sin\theta) \\ y &= a(1 - \cos\theta) \end{aligned}$$

for some $a > 0$ and when θ ranges from 0 to 2π.



3. Evaluate $\iint_S \text{curl } \mathbf{F} \cdot \text{d}\mathbf{S}$, where $\mathbf{F} = \frac{1}{3}\left(-x, \frac{-1}{2}y, x^y\right)$, and S is the union of two pieces S_1 and S_2. S_1 is the half ellipsoid $x^2 + \left(\frac{y}{2}\right)^2 + \left(\frac{z-2}{3}\right)^2 = 1$ and $z \geq 2$. S_1 is the elliptical cylinder $x^2 + \left(\frac{y}{2}\right)^2 = 1$ and $0 \leq z \leq 2$.

4. Evaluate $\iint_S \text{curl } \mathbf{F} \cdot \text{d}\mathbf{S}$, where $\mathbf{F} = (x - z\cos(yz), -y\cos(yz), e^z(1 + x^2))$, and S is the chocolate shell of an ice cream cone that has been completely dipped in chocolate fudge (i.e. every portion of the ice cream cone has been covered).

5. Is $\mathbf{F} = (\cos(xy) - xy\sin(xy), -x^2\sin(xy))$ a gradient vector field? If so, find a scalar potential for $\mathbf{F}$. If not, say why not.

6. Evaluate $\iint_S (x^3, x^3, z^3) \cdot \text{d}\mathbf{S}$ where S is the cylinder (including 'top' and 'bottom') $x^2 + z^2 = 1$ and $0 \leq y \leq 3$ together with the 'top' $x^2 + z^2 \leq 1$ and $y = 0$ and 'bottom' $x^2 + z^2 \leq 1$ and $y = 3$.

7. Evaluate $\iint_E y + e^{-z} + \sin x \; dx \; dy \; dz$ over the region E bounded by the planes $x = 0, y = 0, z = 0$, and $x + y + z = 1$. (Hint: $\int_0^1 (1 - x)^2 \sin x \; dx = 2\cos(1) - 1$.)

8. Sphere A is centered at the origin and the point $(2, 0, 1)$ lies on it. Sphere B is given by the equation $x^2 + y^2 + z^2 = 3$. Which of the following is true?

   (a) A encloses B

   (b) B encloses A

   (c) A and B are equal

   (d) None of the above

9. Level surfaces of the function $f(x, y, z) = (2x^2 + 2y^2)^{1/2}$ are

   (a) Spheres centered at the origin.

   (b) Circles centered at the origin.

   (c) Ellipses (not circles) centered at the origin.

   (d) Upper halves of spheres centered at the origin.

   (e) None of the above.

10. Let $f(3, 5) = 6$, $\frac{\partial f}{\partial x}(3, 5) = -2$, and $\frac{\partial f}{\partial y}(3, 5) = 3$. Then the tangent plane to the surface $z = f(x, y)$ at the point $(3, 5)$ is

    (a) $z - 2x + 3y = 6$

(b) $z + 2x - 3y = -6$

(c) $z + 2x - 3y = -3$

(d) $z + 3x - 2y = 3$

11. Let $(\cos(at), \sin(at))$ be the position at time t seconds of a particle moving around a circle, where $a > 0$. If a is increased,

    (a) The radius of the circle increases.

    (b) The center of the circle changes.

    (c) The path ceases to be a circle.

    (d) The speed of the particle increases.

    (e) The direction of the particle changes.

12. If C_1 is the path parameterized by $\mathbf{c}_1(t) = t\mathbf{i} + t\mathbf{j}$, for $0 \leq t \leq 1$, and if C_2 is the path parameterized by $\mathbf{c}_2(t) = \sin(t)\mathbf{i} + \sin(t)\mathbf{j}$ for $0 \leq t \leq 1$ (radians), and if $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$, which of the following is true?

    (a) $\int_{C_1} \mathbf{F} \cdot d\mathbf{s} > \int_{C_2} \mathbf{F} \cdot d\mathbf{s}$

    (b) $\int_{C_1} \mathbf{F} \cdot d\mathbf{s} < \int_{C_2} \mathbf{F} \cdot d\mathbf{s}$

    (c) $\int_{C_1} \mathbf{F} \cdot d\mathbf{s} = \int_{C_2} \mathbf{F} \cdot d\mathbf{s}$

13. Mark each of the following quantities as **CS** (constant scalar), **SF** (scalar function), **CV** (constant vector), **VF** (vector field), or **ND** (not defined). No reasons need be given. Assume:

    $\mathbf{u}$ is a fixed unit vector in $\mathbb{R}^3$

    $\mathbf{v}$ is a fixed vector in $\mathbb{R}^3$

    $g(x, y, z)$ is a fixed but arbitrary differentiable function whose domain is $\mathbb{R}^3$.

    $\mathbf{F}(x, y, z)$ is a fixed but arbitrary differentiable vector field $\mathbb{R}^3 \to \mathbb{R}^3$

    C is the line from the point $(0, 0, 0)$ to $(5, 6, 7)$

    S is the surface of the unit sphere, centered at the origin and oriented outward

    B is the region inside the unit sphere, S.

    ____ $\mathbf{u} \cdot \mathbf{v}$

    ____ $\mathbf{u} \times \mathbf{v}$

____ $(\mathbf{u} \cdot \mathbf{v})\mathbf{v}$

____ the directional derivative of g int he $\mathbf{u}$ direction, based at the point $(1, 2, 3)$.

____ ∇g

____ $\nabla \mathbf{F}$

____ div g

____ div $\mathbf{F}$

____ curl g

____ curl $\mathbf{F}$

____ curl $\mathbf{F} + \nabla g$

____ $\int_C \nabla g \cdot \mathrm{d}\mathbf{s}$

____ $\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}$

____ $\iiint_B \text{div } \mathbf{F} dV$

____ $\iiint_B \text{curl } \mathbf{F} \cdot \mathrm{d}\mathbf{S}$

14. (a) Sketch or describe a graph of the function $f(x, y) = x^2 - y^2$.
    (b) Is $\frac{\partial^2 f}{\partial x^2}(0, 0)$ positive or negative?
    (c) Is $\frac{\partial^2 f}{\partial x \partial y}(0, 0)$ positive or negative?

15. Let B be half of a cylinder of radius 3 and height 5. Write 3 triple integrals to calculate (but do not calculate) the volume of B, one for each of cartesian, cylindrical, and spherical coordinates.

16. (a) State the Divergence Theorem.
    (b) Find the flux of the vector field $\mathbf{F} = x^2\mathbf{i} + (z - 2xy)\mathbf{j} + zx\mathbf{k}$ out of the sphere of radius 5 centered at the origin.

17. (a) State Stoke's Theorem.
    (b) Let C be the circle of radius 3 in the xy-plane, centered at the origin, $\mathbf{r}(x, y, z) = (x, y, z)$, and $\mathbf{F} = \frac{\mathbf{r}}{\|\mathbf{r}\|}$. Explain why Stoke's Theorem is a better choice than Green's Theorem to solve $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{s}$.
    (c) Evaluate $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{s}$. (Stoke's theorem might not be the easiest method).

18. Let S be the surface given by the graph of $z = x^2 + y^2$ that lies within the cylinder $x^2 + y^2 = 9$. Give S the outward orientation.

(a) Find the area of S.

(b) A fluid with velocity field $\mathbf{F} = -y\mathbf{i} + x\mathbf{j} + 2z\mathbf{k}$ flows through S. Find the rate of flow through the surface S.

19. Let S be the hemisphere defined by $x^2 + y^2 + z^2 = 1, y \geq 0$, oriented in the positive y direction. Let $\mathbf{F} = 2z\mathbf{i} - x\mathbf{j} + \sin^2(y)\mathbf{k}$.

    (a) Evaluate $\displaystyle\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$.

    (b) Is $\mathbf{F}$ a gradient vector field? Explain your answer.

20. Let $f(x, y) = e^{x^2} \sin(y) - y + 1$. Find the equation of the plane tangent to $z = f(x, y)$ at the point $(1, \frac{\pi}{2})$.

21. Calculate the flux of $\mathbf{F} = (x^3 + y \sin z)\mathbf{i} + (y^3 + z \sin x)\mathbf{j} + z^3\mathbf{k}$ across the closed surface bounded by the hemispheres $z = \sqrt{4 - x^2 - y^2}$, $z = \sqrt{1 - x^2 - y^2}$ and the plane $z = 0$.

22. Consider the following two vector fields in $\mathbb{R}^3$:

$$\mathbf{F} = y^2\mathbf{i} - z^2\mathbf{j} + x^2\mathbf{k}\,, \quad \mathbf{G} = (x^3 - 3xy^2)\mathbf{i} + (y^3 - 3x^2y)\mathbf{j} + z\mathbf{k}\,.$$

    (a) Which of these fields are conservative? Give reasons for your answer.

    (b) For each field that is conservative, find a function f such that ∇f gives the field.

23. Let W be a solid region obtained as follows. Start with a solid ball of radius 3, centered at the origin, and delete the region within the double cone $z^2 = x^2 + y^2$.

    (a) Find the volume of W.

    (b) Suppose W has a mass density function given by $\delta(x, y, z) = 10 - z$. Find the total mass of W.

    (c) Let S be the boundary of W, oriented outward, and let $\mathbf{F}$ be a vector field which is continuously differentiable on all of $\mathbb{R}^3$. What can you say about $\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$?

24. Evaluate $\displaystyle\int_C \big(\cos(x)\mathbf{i} + 3x\mathbf{j}\big) \cdot d\mathbf{s}$, where C is the triangle with vertices $(-1, 0), (0, 2)$, and $(3, 0)$, oriented clockwise.