MATH 365 Homework #12 due December 7 FALL 2007

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book problems

§11.3 #4(b). Show there are infinitely may even abundant numbers. As per the hint, consider $n = 2^k \cdot 3$. $\sigma(n) = (2^{k+1} - 1)4$. We need to show that this is bigger than $2^{k+1}3 = 2n$. This follows because $2^{k-1} - 1 > 0$ by our assumption that $k > 1$. Add $3 \cdot 2^{k-1}$ to both sides. We get $4 \cdot 2^{k-1} - 1 = 2^{k+1} - 1 > 3 \cdot 2^{k-1}$. Multiplying both sides by 4 we get $\sigma(n) = (2^{k+1} - 1)4 > 3 \cdot 2^{k+1}$.

§11.4 #17(a). Note that $F_n = 2^{2^n} + 1 \equiv 1 \pmod{4}$ for all $n \geq 2$. Therefore, $(3/F_n) = (F_n/3)$. Now, $2^{2^n} + 1 \equiv (-1)^{2^n} + 1 \equiv -1 \pmod{3}$. Therefore $(3/F_n) = (-1/3) = -1$. To compute $(5/F_n)$ note that $2^{2^n} + 1 = 4^{2^{n-1}} + 1 \equiv (-1)^{2^{n-1}} + 1 \equiv 1 + 1 \equiv 2 \pmod{5}$. Therefore, $(5/F_n) = (F_n/5) = (2/5) = -1$. By problem 15, section 9.3, since F_n is of the form $p = 2^{4n} + 1$, we see that 7 is a primitive root of F_n. Therefore, the order of $7 \pmod{F_n}$ is $F_n - 1$ and therefore $7^{(F_n-1)/2} \equiv -1 \pmod{F_n}$. (Since $7^{(F_n-1)/2}$ satisfies $x^2 \equiv 1 \pmod{F_n}$ and $7^{(F_n-1)/2}$ is not equivalent to $1 \pmod{F_n}$, we must have $7^{(F_n-1)/2} \equiv= 1 \pmod{F_n}$.

Non-book problems:

1. We proceed by induction. Case $n = 0$ comes down to showing $3 = F_0 = F_1 - 2$. But this is true since $F_1 = 5$. Assume by induction $\prod_{i=0}^{k-1} = F_k - 2$. We need to show that $\prod_{i=0}^{k} F_i = F_{k+1} - 2$. By the induction hypothesis this boils down to showing $(F_k - 2)F_k = F_{k+1} - 2$ which follows quickly.

2. Let q be a prime dividing $\prod_{i=0}^{n} F_i$. Claim: q does not divide F_{n+1}. Proof of claim: If $q \mid F_{n+1}$, then $q \mid (F_{n+1} - F_0F_1 \dots F_n)$ and therefore $q \mid 2$ by part 1. This is a contradiction since all F_i are odd. Therefore $q \nmid F_{n+1}$. Let $T_n = \{\text{primes dividing} \prod_{i=0}^{n} F_i\}$. Then each T_n is a proper subset of T_{n+1}. In particular, as n goes to infinity, our list of primes is infinitely long.