book problems

§2.3

#18 Let $k, k+1, k+2$ be three consecutive integers. By the division algorithm, one of them is divisible by 3, hence $3 \mid k(k+1)(k+2)$. By a similar argument, $2 \mid k(k+1)(k+2)$. Hence corollary 2 to Theorem 2.4 shows $6 \mid k(k+1)(k+2)$.

Let $k, k+1, k+2, k+3$ be four consecutive integers. Then, k is of the form $k = 4j + i$ for some $j \in \mathbb{Z}$ and some $i = 0, 1, 2, 3$. Substituting in $4j + i$ for k we see that one of the set $\{k, k+1, k+2, k+3\}$ is divisible by 4 and a different element of the set is divisible by 2, hence $8 \mid k(k+1)(k+2)(k+3)$. Also we have $3 \mid k(k+1)(k+2)(k+3)$ by the division algorithm. Therefore, $24 \mid k(k+1)(k+2)(k+3)$.

Let $k, k+1, k+2, k+3, k+4$ be five consecutive integers. By the previous argument, $24 \mid k(k+1)(k+2)(k+3)(k+4)$. Also, $5 \mid k(k+1)(k+2)(k+3)(k+4)$ by the division algorithm. Therefore, $5 \cdot 24 = 120 \mid k(k+1)(k+2)(k+3)(k+4)$

§2.4

#6 Show: if $\gcd(a, b) = 1$, then $\gcd(a+b, ab) = 1$.
Set $d = gcd(a+b, ab)$. Since $d \mid a+b$ and $d \mid ab$ we have,

$$\begin{aligned} d &\mid [ab - (a+b)^2] \\ d &\mid [-a^2 - b^2 - ab] \\ d &\mid -(a(a+b) + b^2) \\ d &\mid -b^2 \end{aligned}$$

Since we've shown $d \mid -b^2$ and $d \mid ab$, we have, $d \mid \gcd(-b^2, ab)$ and therefore $d \mid b\gcd(-b, a) = b$. An analogous procedure shows that $d \mid b$. Therefore $d \mid \gcd(a, b) = 1$. Therefore $d = 1$.