book problems

§6.1 12 a). According to the formula, if $n = \prod_{i=1}^{r} p_i^{k_i}$, then $\tau(n) = \prod_{i-1}^{r}(k_i + 1)$. Therefore, if $\tau(n) = 10$ we have only a limited number of possibilities. 10 can be written as a product in only two ways, $10 = 10 \cdot 1$ or $10 = 2 \cdot 5$. Therefore, the possibilities for n are $n = p_1^9$, $n = p_1^1 p_2^4$, where p_1 and p_2 are two distinct primes.
b). As noted in the hint, $\sigma(n) > n$ as long as $n > 1$. This is because $\sigma(n) = \sum_{d|n} d$ and hence for $n \neq 1$, $\sigma(n) \geq n+1$ (because both 1 and n are guaranteed to be divisors of n). So, to prove that $\sigma(n) \neq 10$ for all n, we need only show $\sigma(x) \neq 10$ for $x = 1, 2, 3, 4, 5, 6, 7, 8, 9$. Either doing it by hand, or punching it into a computer we see that the values of $\sigma(x)$ are $1, 3, 4, 7, 6, 12, 8, 15, 13$ as x runs from 1 to 9.

§6.2 1 a).Given any four consecutive integers, one of them is divisible by 4. Since μ is multiplicative $\mu(4m) = 0$ for any positive integer m. Therefore, at least one of $\mu(n)$, $\mu(n+1)$, $\mu(n+2)$ or $\mu(n+3)$ is zero and hence when you multiply all of them together, you get zero.
b). By part (a), $\mu(k!) = 0$ when $k \geq 4$. Therefore, for $n \geq 3$, $\sum_{k=1}^{n} \mu(k!) = \mu(1!) + \mu(2!) + \mu(3!) = 1 + (-1) + (1) = 1$.

§7.2 1 $\phi(1001) = 720$, $\phi(5040) = 1152$, $\phi(36000) = 9600$.

§7.2 4 a). Since $\gcd(n, 2) = 1$, $\phi(2n) = \phi(2)\phi(n) = 2(1 - 1/2)\phi(n) = \phi(n)$.
b). Set $n = 2^k \cdot q$ with $\gcd(2, q) = 1$. Then $\phi(2n) = \phi(2^{k+1}q) = \phi(2^{k+1})\phi(q) = 2\phi(2^k)\phi(q) = 2\phi(n)$.
e).Let $n = 2^k \cdot N$ with $\gcd(2, N) = 1$. Then $\phi(n) = \phi(2^k)\phi(N) = 2^{k-1}\phi(N)$. By assumption this equals $n/2 = 2^{k-1}N$. Dividing both sides by 2^{k-1} we see that $\phi(N) = N$ and this happens only for $N = 1$.

Non-book problems:

$M(659) = -10$ $M(1393) = 10$
$M(59426) = -10^2$ $M(114718) = 10^2$
$M(7102866) = -10^3$ $M(6447543) = 10^3$