§7.2 #12. Let $\mathbf{F} = (z^3 + 2xy)\mathbf{i} + x^2\mathbf{j} + 3xz^2\mathbf{k}$. Show that the integral of $\mathbf{F}$ around the circumference of the unit square with verticies $(\pm 1, \pm 1)$ is zero. Let $\mathbf{c} : [a, b] \to \mathbb{R}^3$ be a path parameterizing the described square. Since the square is a closed path (begins and ends in the same spot), we have $\mathbf{c}(a) = \mathbf{c}(b)$. Notice that $\mathbf{F} = \nabla f$, where $f(x, y, z) = xz^3 + x^2y$. Therefore, according to Theorem 3 in section 7.2,

$$\int_{\mathbf{c}} \mathbf{F}.d\mathbf{s} = \int_{\mathbf{c}} \nabla f.d\mathbf{s} = f(\mathbf{c}(b)) - f(\mathbf{c}(a)) = 0.$$

The last equality is because $\mathbf{c}(a) = \mathbf{c}(b)$.

§7.2 #16. Suppose $\nabla f(x, y, z) = 2xyze^{x^2}\mathbf{i} + ze^{x^2}\mathbf{j} + ye^{x^2}\mathbf{k}$. If $f(0, 0, 0) = 5$, find $f(1, 1, 2)$. Let $\mathbf{c} : [0, 1] \to \mathbb{R}^3$ be the path given by $\mathbf{c}(t) = (t, t, 2t)$. Then $\mathbf{c}(0) = (0, 0, 0)$ and $\mathbf{c}(1) = (1, 1, 2)$. By Theorem 3 in section 7.2 we know

$$\int_{\mathbf{c}} \nabla f.d\mathbf{s} = f(1, 1, 2) - f(0, 0, 0) = f(1, 1, 2) - 5.$$

Therefore we need to calculate $\int_{\mathbf{c}} \nabla f.d\mathbf{s}$.

$$\int_{\mathbf{c}} \nabla f.d\mathbf{s} = \int_0^1 \nabla f.\mathbf{c}'(t)\, dt = 4 \int_0^1 (t^3 + t)e^{t^2}\, dt = 2e.$$

Therefore, $f(1, 1, 2) - 5 = 2e$. Solving for $f(1, 1, 2)$ we get $f(1, 1, 2) = 2e + 5$.

§7.3 #4. A surface is regular when $T_u \times T_v \neq \mathbf{0}$. For problem 1 we calculate $T_u = (2, 2u, 0)$ and $T_v = (0, 1, 2v)$. Therefore, $T_u \times T_v = 4uv\mathbf{i} - 4v\mathbf{j} + 2\mathbf{k}$. This vector is never the $\mathbf{0}$ vector because the final component is always 2. For problem 2 we calculate $T_u = (2u, 1, 2u)$ and $T_v = (-2v, 1, 4)$. Therefore, $T_u \times T_v = (4 - 2u)\mathbf{i} - (8u + 4uv)\mathbf{j} + (2u + 2v)\mathbf{k}$. This vector is zero when $u = -v = 2$. Therefore the surface is not regular at $(0, 0, -4)$.