Solutions to Homework #14 from section 3.4. Please email me if you detect any errors in these solutions.

2. Find extrema of $f(x,y) = x - y$ subject to $g(x,y) = x^2 - y^2 = 2$.
   Solution. Checking to see when $\nabla g = (2x, -2y) = (0,0)$ we see that $x = 0$ and $y = 0$. Since this is not on the curve $x^2 - y^2 = 2$, we do not have to worry about this point. Using the method of Lagrange Multipliers, we must solve the equations

$$\begin{aligned} \lambda &= 2x \\ -\lambda &= -2y \end{aligned}$$

   In other words, $x = \frac{\lambda}{2} = y$. But, putting this into our constraining equation we see that $x^2 - y^2 = 0$ since this is not 2, there is no point where the gradient vectors are parallel. Therefore, no minimum or maximum of $f(x,y)$ occurs on the (hyperbola) $x^2 - y^2 = 2$.

3. Find extrema of $f(x,y) = x$ subject to $g(x,y) = x^2 + 2y^2 = 3$.
   Solution. Check $\nabla g = (2x, 4y)$. This equals $(0,0)$ only if $x = y = 0$ which is not on the curve $x^2 + 2y^2 = 3$. As in problem 2 we must solve the equations

$$\begin{aligned} \lambda &= 2x \\ 0 &= 4y \end{aligned}$$

   This gives us $y = 0$ and $x = \frac{2}{\lambda}$. Plugging this into $g(x,y) = 3$ we see that $x = \pm\sqrt{3}$. Therefore the max is at $(\sqrt{3}, 0)$ and the min is at $(-\sqrt{3}, 0)$.

5. Find extrema of $f(x,y) = 3x + 2y$ subject to $g(x,y) = 2x^2 + 3y^2 = 3$.
   Solution. Check $\nabla g = (4x, 6y)$. This equals $(0,0)$ only if $x = y = 0$ which is not on the curve $g(x,y) = 3$. As in problem 2 we must solve the Lagrange Multiplier equations

$$\begin{aligned} 3\lambda &= 4x \\ 2y &= 6y \end{aligned}$$

   This gives us $x = \frac{3}{4}\lambda$ and $y = \frac{1}{3}\lambda = \frac{4}{9}x$. Plugging this into the constraining equation $g(x,y) = 3$ we find that the possible solutions are

$$\left(\frac{9}{\sqrt{70}}, \frac{4}{\sqrt{70}}\right) \text{ where the max occurs, and } \left(-\frac{9}{\sqrt{70}}, -\frac{4}{\sqrt{70}}\right) \text{ where the min occurs.}$$

10. Use the method of Lagrange Multipliers to find the absolute max and min values of $f(x,y) = x^2 + y^2 - x - y + 1$ on the unit disk.
    Solution. We first find the places that the extrema occur on the boundary of the disk $x^2 + y^2 = 1$. Using the method of Lagrange Multipliers we need to solve the equations

$$\begin{aligned} \lambda x &= 2x - 1 \\ \lambda y &= 2y - 1 \end{aligned}$$

Solving these for x and y we see $x = \frac{1}{\lambda-2} = y$. Putting this into the equation $x^2+y^2 = 1$ we see that $x = y = \pm\frac{1}{\sqrt{2}}$. Therefore the possible min and max of $f(x, y)$ that occur on the boundary are $f(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = 2 - \sqrt{2}$ and $f(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = 2 + \sqrt{2}$. To check the possible min and max in the interior of the circle we look for critical points: $\nabla f(x, y) = (2x - 1, 2y - 1) = (0, 0)$ when $x = y = \frac{1}{2}$. Since $(\frac{1}{2}, \frac{1}{2})$ is in the interior of the unit circle it is a possible min/max. We see that $f(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}$. Since $\frac{1}{2} < 2 - \sqrt{2} < 2 + \sqrt{2}$, the absolute minimum occurs at $(\frac{1}{2}, \frac{1}{2})$ and the absolute max occurs at $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$.

13. Let r be the radius of the can and let h be the height. According to the stated problem, we need to minimize $f(r, h) = 2\pi(r^2 + rh)$ which is the surface area of the can, with the constraint $g(r, h) = \pi r^2 h = 1000\text{cm}^3$, which says that the volume of the can is 1000cm^3.
Solution. According to the method of Lagrange Multipliers, the max or min, (r, h), occurs at a point with $\nabla g(r, h) \neq (0, 0)$, $\nabla f(r, h) = \lambda\nabla g(r, h)$ and $g(r, h) = 1000$. Since $\nabla g(r, h) = (2\pi rh, \pi r^2) = (0, 0)$ only if $r = h = 0$, we do not have to worry about this constraint since $g(0, 0) \neq 1000$. The equation $\nabla f(r, h) = \lambda\nabla g(r, h)$ is solved by setting

$$\begin{aligned} 2r + h &= \lambda 2rh \\ r &= \lambda r^2 \end{aligned}$$

(In calculating this we cancelled a π from each side and absorbed 2 into the constant λ). By the second equation $r(1 - \lambda r) = 0$. This happens if $r = 0$ (which is impossible, because the volume would be 0), or if $r = \frac{1}{\lambda}$. Plugging $r = \frac{1}{\lambda}$ into the first equation we see $h = \frac{2}{\lambda} = 2r$. Plugging $h = 2r$ into $g(r, h) = 1000$ we see that

$$r = \frac{10}{(2\pi)^{1/3}} \qquad \text{and} \qquad h = \frac{20}{(2\pi)^{1/3}}$$

Therefore the can with minimum surface area has this radius and height. To check that this is a minimum you could compute the surface area at any other (r, h) satisfying $g(r, h) = 1000$. For example $(1, \frac{1000}{\pi})$ satifies $g(1, \frac{1000}{\pi}) = 1000$ and $f(1, \frac{1000}{\pi}) = 2006.3$, while $f(\frac{10}{(2\pi)^{1/3}}, \frac{20}{(2\pi)^{1/3}}) = 553.6$