

§2.2 Limits and Continuity

Caution: It is NOT enough to prove that a limit is a specific number just by showing that the limit is that number if you approach it from different directions. Many people made this mistake on #8(a) and #8(c). Here are the correct solutions.

8a.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{(x+y)^2 - (x-y)^2}{xy}$$

Multiplying out the top of this fraction, we see that this is the same as

$$\lim_{(x,y) \rightarrow (0,0)} \frac{4xy}{xy} = \lim_{(x,y) \rightarrow (0,0)} 4 = 4.$$

Some people made the mistake of setting one of x or y equal to 0, and then saying the limit D.N.E. This does not work because $x = 0$ and/or $y = 0$ are not in the domain of the function. $(x, y) \rightarrow (0, 0)$ means that x and y approach $(0, 0)$ along paths in the domain of the function! It is only the limit point (in our case $(0, 0)$) which is allowed to be in the boundary of the domain.

8c.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^3 - y^3}{x^2 + y^2}$$

Note that $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$. Furthermore, notice that $|xy| \leq x^2 + y^2$, or in other words, $0 \leq x^2 + xy + y^2 \leq 2(x^2 + y^2)$. This leads to the inequality

$$0 \leq \frac{x^2 + xy + y^2}{x^2 + y^2} \leq \frac{2(x^2 + y^2)}{x^2 + y^2} = 2$$

We now have the bound

$$0 \leq \lim_{(x,y) \rightarrow (0,0)} \frac{|x^3 - y^3|}{x^2 + y^2} \leq \lim_{(x,y) \rightarrow (0,0)} |x - y|2 = 0$$

Therefore, our limit is zero!

Be careful! On the homework, many people saw that this limit was zero in a couple of experiments (for example set $x = 0$ or $y = 0$ or $y = x$), and decided that the answer was zero based on these experiments. Even though the answer was correct, this does not prove it! Also, some people said $x^2 + y^2$ divides $x^3 - y^3$, which is not true. Some other people changed the problem to polar coordinates which seemed to work well too.