§2.5 Properties of the Derivative.

#5Verify the first special case of the chain rule for the composition $f \circ \mathbf{c}$ in each of the cases:

a. $f(x,y) = xy$, $\mathbf{c}(t) = (e^t, \cos t)$. In order to verify this, we just need to compute the derivative of $f \circ \mathbf{c}$ in two ways. On the one hand, $f \circ \mathbf{c}(t) = e^t \cos t$ and we can easily compute its derivative to be $\frac{d}{dt}(f \circ \mathbf{c}) = e^t \cos t - e^t \sin t$. On the other hand by the chain rule we have

$$D(f \circ \mathbf{c}) = Df(c(t))Dc(t).$$

Where $Df = [f_x, f_y] = [y, x]$, $Df(c(t)) = [\cos t, e^t]$ and $D\mathbf{c} = \begin{bmatrix} e^t \\ -\sin t \end{bmatrix}$. Therefore by the chain rule,

$$D(f \circ \mathbf{c}) = [\cos t, e^t] \begin{bmatrix} e^t \\ -\sin t \end{bmatrix} = e^t \cos t - e^t \sin t.$$

This verifies the chain rule in this case. You could also recognize the chain rule in this case as

$$D(f \circ \mathbf{c})(t) = f_x \cdot \frac{dx}{dt} + f_y \cdot \frac{dy}{dt}.$$

Then computing $f_x = y$, $f_y = x$, $\frac{dx}{dt} = e^t$ and $\frac{dy}{dt} = -\sin t$, you would end up with the exact same answer.

c. $f(x,y) = (x^2 + y^2)\log\sqrt{x^2+y^2}$, $\mathbf{c}(t) = (e^t, e^{-t})$. In this problem $\log = \ln$, the natural logarithm. Computing $f \circ \mathbf{c}$ we get $(f \circ \mathbf{c})(t) = (e^{2t} + e^{-2t})\log\sqrt{e^{2t}+e^{-2t}}$. Computing the derivative we see

$$\begin{aligned} (f \circ \mathbf{c})'(t) &= (2e^{2t} - 2e^{-2t})\log\sqrt{e^{2t}+e^{-2t}} + (e^{2t}+e^{-2t})\frac{(2e^{2t}-2e^{-2t})}{2(e^{2t}+e^{-2t})} \\ &= (2e^{2t} - 2e^{-2t})\log\sqrt{e^{2t}+e^{-2t}} + (e^{2t}-e^{-2t}) \\ &= (e^{2t}-e^{-2t})(2\log\sqrt{e^{2t}+e^{-2t}} + 1) \end{aligned}$$

On the other hand, by the chain rule,

$$\begin{aligned} D(f \circ \mathbf{c})(t) &= Df(c(t))Dc(t) \\ &= \left[2x\log\sqrt{x^2+y^2} + x, 2y\log\sqrt{x^2+y^2} + y\right] \begin{bmatrix} e^t \\ -e^{-t} \end{bmatrix} \\ &= (2e^t\log\sqrt{e^{2t}+e^{-2t}} + e^t)e^t - (2e^{-t}\log\sqrt{e^{2t}+e^{-2t}} + e^{-t})e^{-t} \\ &= (e^{2t}-e^{-2t})(2\log\sqrt{e^{2t}+e^{-2t}} + 1) \end{aligned}$$

#8. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be differentiable. Using spherical coordinates, compute f_ρ, f_θ and f_ϕ in terms of f_x, f_y and f_z.

$$f(x,y,z) = f(\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi).$$

This is the same as taking the composition of the two function $f \circ c$, where $c(\rho, \theta, \phi) = (\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)$. The chain rule tell us:

$$D(f \circ c) = Df(c)\, Dc$$

Here $Df = [f_x, f_y, f_z]$ and

$$Dc = \begin{bmatrix} \cos\theta\sin\phi & -\rho\sin\theta\sin\phi & \rho\cos\theta\cos\phi \\ \sin\theta\sin\phi & \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \\ \cos\phi & 0 & -\rho\sin\phi \end{bmatrix}$$

Therefore, $D(f \circ c) = [f_\rho, f_\theta, f_\phi]$ is gotten by

$$\begin{aligned} f_\rho &= f_x\cos\theta\sin\phi + f_y\sin\theta\sin\phi + f_z\cos\phi \\ f_\theta &= -f_x\rho\sin\theta\sin\phi + f_y\rho\cos\theta\sin\phi \\ f_\phi &= f_x\rho\cos\theta\cos\phi + f_y\rho\sin\theta\cos\phi - f_z\rho\sin\phi \end{aligned}$$

#13. The position of the duck at time t is $p(t) = (\cos t, \sin t)$. The temperature of the water is $T(x, y) = x^2e^y - xy^3$.

a. Find $\frac{dT}{dt}$ using the chain rule. By the chain rule,

$$\begin{aligned} \frac{dT}{dt} &= T_x\frac{dx}{dt} + T_y\frac{dy}{dt} \\ &= (2xe^y - y^3)(-\sin t) + (x^2e^y - 3xy^2)(\cos t) \\ &= (2\cos te^{\sin t} - \sin^3 t)(-\sin t) + (\cos^2 te^{\sin t} - 3\cos t\sin^2 t)(\cos t). \end{aligned}$$

b. Find $\frac{dT}{dt}$ by expressing T in terms of t and differentiating.

$$T(t) = \cos^2 te^{\sin t} - \cos t\sin^3 t.$$

$$T'(t) = -2\cos t\sin te^{\sin t} + \cos^3 te^{\sin t} + \sin^4 t - 3\cos^2 t\sin^2 t$$

You should check that these two answers are the same.