Week 1, Day 3

5/29/08

1 Intersection multiplicities in general

- We say that f and g *intersect properly* at P if f and g have no common component that passes through P. Let f and g be two curves which intersect at the origin. We can compute the **intersection multiplicity** of f and g at $\mathbf{O} = (0,0)$, denoted $I_{\mathbf{O}}(f,g)$ following the these six properties:

  1. $I_{\mathbf{O}}(f,g)$ is a nonnegative integer for any f, g such that f and g intersect properly at $\mathbf{O}$. $I_{\mathbf{O}}(f,g) = \infty$ if f and g do not intersect properly at $\mathbf{O}$.
  2. $I_{\mathbf{O}}(f,g) = I_{\mathbf{O}}(g,f)$
  3. $I_{\mathbf{O}}(f,g) = 0$ if and only if $\mathbf{O} \notin \mathbf{f} \cap \mathbf{g}$.
  4. $I_{\mathbf{O}}(x,y) = 1$
  5. $I_{\mathbf{O}}(f,g) = I_{\mathbf{O}}(f, g + fh)$ for any polynomial $h \in \mathbb{C}[x,y]$.
  6. $I_{\mathbf{O}}(f,gh) = I_{\mathbf{O}}(f,g) + I_{\mathbf{O}}(f,h)$

  properties:

  7. If f and g are polynomial such that f is a factor of g and the curve $f = 0$ contains the origin $\mathbf{O}$, then $I_{\mathbf{O}}(f,g)$ is ∞.
  8. If f, g and h are curves and g does not contain the origin, we have

  $$I_{\mathbf{O}}(f,gh) = I_{\mathbf{O}}(f,h).$$

- Example: How many times do the curves $y^3 + 2x^5 = 0$ and $xy^2 + y - 3x^3 = 0$ intersect at the origin? (Bix pg. 13).

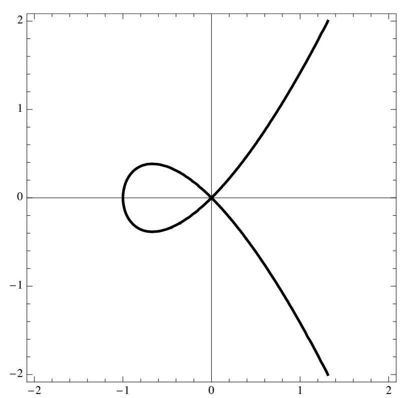
- **Rational Parametrization** Another way to give a curve is via a parametrization. E.g., the set of points (t^2, t^3) for $t \in C$ are the solutions to $y^2 - x^3 = 0$. A curve with a rational parametrization is seen as especially nice. Formal definition: An irreducible algebraic curve C defined by $f(x,y) = 0$ is *rational* if there exist two rational functions $\varphi(t)$ and $\psi(t)$, at least one non-constant, such that

$$f(\varphi(t), \psi(t)) = 0,$$

as an identity in t. Note that if $t = t_0$ is a value of t, and is not one of the finitely many values at which the denominator of φ or ψ vanishes, then $(\varphi(t), \psi(t))$ is a point of C.

- Rational parametrization. In vector calc you learned how to parametrize curves (and surfaces even). Implicit form is in both x and y. A **rational curve** f is one where there exists two functions $x(t) = \frac{p_1(t)}{q_1(t)}$ and $y(t) = \frac{p_2(t)}{q_2(t)}$ (at least one nonconstant) so that p_i and q_i are polynomials in t and $f(x(t), y(t)) = 0$ for all t. In otherwords, $x(t)$ and $y(t)$ trace out our curve. e.g., an easy one: $y = x^2$ can be parametrized by (t, t^2) (can do this for any function). But it gets harder: $y^2 - x^3$ can be parametrized by (t^2, t^3).
  This gives us a map from $\mathbb{C}$ to our curve, except at the points where $q_i(t) = 0$. But almost everywhere! So we can think of rational curves as very similar to $\mathbb{C}$.

- Example: $y^2 = x^2 + x^3$. This looks like: Consider the line through the origin $y = tx$.



Look for the intersection of this line with our curve: $x^2(t^2 - x - 1) = 0$; the double root corresponds to the origin. The other root is given by $t^2 - 1 = x$, implying that $y = t(t^2 - 1)$. This is our parametrization. Notice that it is defined everywhere, this will not always be the case! And, this method might not always work either!