

Exercises 5/27/08

Please attempt the following problems. Use of computers, friends, and myself are encouraged.

1. Find the singular points of these curves by taking partial derivatives. Next find the Taylor expansion of the curve about the singular points you found. If the lowest degree terms in the Taylor expansion are of degree 2, then determine if the point is a node or a cusp. Finally use a computer package (Maple or Mathematica) to plot the curves using “implicit plot” in Maple, or “ContourPlot” in Mathematica.

a. $x^2 = x^4 + y^4$

b. $xy = x^6 + y^6$

c. $x^3 = y^2 + x^4 + y^4$

d. $x^2y + xy^2 = x^4 + y^4$

e. $y^2 = x^2 + x^3$

f. $(x^2 + y^2)^2 = x^2 - y^2$

2. Prove the following statements.

a. Any non-zero complex form $f \in \mathbb{C}[x, y]$ can be written uniquely (up to order) in the form $f(x, y) = (\alpha_1x + \beta_1y)^{r_1} \dots (\alpha_sx + \beta_sy)^{r_s}$ where the ratios $(\alpha_1 : \beta_1), \dots, (\alpha_s : \beta_s)$ are uniquely defined.

b. Let F, G be non-zero forms in $\mathbb{C}[x, y]$ of degrees $d, d+1$ respectively, having no common factors. Then $f = F + G$ is irreducible.

3. Factor $Y^3 - 2XY^2 + 2X^2Y + X^3$ into linear factors.

4. Let $V(d, n) = \{\text{forms of degree } d \text{ in } \mathbb{C}[x_1, \dots, x_n]\}$ (n variables). Show that $V(d, n)$ is a vector space over \mathbb{C} , and that the monomials of degree d form a basis. Show that the dimension of $V(d, n)$ is the binomial coefficient $\binom{d+n-1}{n-1}$. Note that in the case we are interested in, $n = 2$, we have $\dim V(d, 2) = d + 1$.

5. (For fun!) Locate the singular points and describe the singularities of the following surfaces in \mathbb{R}^3 .

a. $xy^2 = z^2$

b. $x^2 + y^2 = z^2$

x. $xy + x^3 + y^3 = 0$