Exercises 6/9/08

1. Let k be an algebraically closed field.

   (a) Suppose that f and g are homogeneous polynomials of degree s and t, respectively, that do not share any linear factors. Show that $\langle f, g\rangle_d = k[x,y]_d$ for $d \geq s+t-1$ (that is, f and g generate all the d-th degree polynomials).

   (b) Let $f = f_r + f_{r+1} + f_{r+2} + \ldots \in k[[x,y]]$, where f_k is the k-th degree homogeneous part of f. Suppose that f_r factors as $f_r = g_s h_t$ for homogeneous polynomials g_s and h_t of degrees s and t respectively, and that g_s and h_t share no common linear factor. Show that there are formal power series $g, h \in k[[x,y]]$ such that $f = gh$.

2. Let Y be defined by the equation $f(x,y) = 0$ in $\mathbb{A}^2$, and let $P = (0,0)$ be a point of multiplicity r on Y, so we may write $f = f_r + \text{h.o.t.}$. A **double point** is a point of multiplicity two where f_2 has two distinct linear factors. More generally, we say that P is an **ordinary r-fold point** of Y if f_r is the product of r distinct linear factors.

   (a) Show that any two ordinary double points are analytically isomorphic.

   (b) Show that any ordinary triple points are analytically isomorphic.

   (c) Show that there is a family of mutually non-isomorphic ordinary 4-fold points.

3. Recall that the Milnor number of a plane curve singularity is defined as

$$\mu_{f,0} = \dim \frac{\mathbb{C}[[x,y]]}{\langle \partial f/\partial x, \partial f/\partial y\rangle}$$

   and the Tjurina number is defined to be

$$\tau_{f,0} = \dim \frac{\mathbb{C}[[x,y]]}{\langle f, \partial f/\partial x, \partial f/\partial y\rangle}$$

   Calculate the Milnor and Tjurina numbers for $f(x,y) = x^3 + y^3 - x^2 y$ at the origin.

4. Calculate the Milnor and Tjurina numbers at the origin for

   (a) $f(x,y) = y^3 - x^7 + x^5 y$.

   (b) $f(x,y) = x^p + y^q$, for integers p and q.

5. The Tjurina and Milnor numbers of a singularity are invariants of the analytic equivalence class of the singularity (that is, if two singularities are analytically equivalent, they have the same Tjurina and Milnor numbers). Sketch a proof of this fact.