Exercises 6/11/08

1. Determine the Puiseux pairs for the curves given in the previous exercises.

2. Euler's Theorem says: Let $F = \sum_{i+j+k=n} a_{ijk}x^i y^j z^k$ be an arbitrary homogeneous pollynomial of degree n. Then,

$$nF = x\frac{dF}{dx} + y\frac{dF}{dy} + z\frac{dF}{dz}.$$

   Prove Euler's Theorem.

3. Let $f^h(x, y, z)$ be the homogenization of $f(x, y)$. Suppose that $V(f)$ is singular at the point $(a, b) \in \mathbb{C}^2$. Prove that $V(f^h)$ is singular at $(a : b : 1) \in \mathbb{P}^2$.

4. Homogenize the following equations and find which points at infinity are added to projectivize the associated curve. Draw the graph and "draw-in" as best you can the points at infinity.

   (a) $y = 2x - 3$
   (b) $2y^3 + 4x + 1 = 0$
   (c) $x^3 = y^2$
   (d) $x^2y + y^3 = x^3$
   (e) $x^2 + 2xy = 1$

5. Find an example of a smooth affine curve which has a singular projectivization.

6. Determine whether any of the following curves have singularities at any points of infinity. If so, determine their multiplicity.

   (a) $(x^2 + y^2)^2 = x^2 - y^2$
   (b) $y^2 = x^2 + x^3$
   (c) $y = x^2y + x^3$

7. Consider the sets $U_i \subseteq \mathbb{P}^n$ given by $U_i = \{\mathbf{p} \in \mathbb{P}^\mathbf{n} : \mathbf{a_i} \neq \mathbf{0}\}$, and the maps

$$\begin{aligned}
&\phi_i : \mathbb{A}^n \to \mathbb{P}^n \\
&\phi_i(a_1, \ldots, a_n) = (a_1 : \ldots : a_{i-1} : 1 : a_i : \ldots : a_n) \\
&\psi_i : U_i \to \mathbb{A}^n \\
&\psi_i(a_0 : \ldots : a_n) = (\frac{a_0}{a_i}, \ldots, \frac{a_{i-1}}{a_i}, \frac{a_{i+1}}{a_i}, \ldots, \frac{a_n}{a_i})
\end{aligned}$$

   Verify that there is a bijection from each set U_i to $\mathbb{A}^n$ by showing that the maps ϕ_i and ψ_i are inverses.

8. Consider the maps

$$\mu_i : k[x_0, \ldots, x_n]_h \to k[y_0, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n]$$
$$\mu_i(f) = f(y_0, \ldots, y_{i-1}, 1, y_{i+1}, \ldots, y_n)$$
$$\nu_i : k[y_0, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n] \to k[x_0, \ldots, x_n]_h$$
$$\nu_i(f) = x_i^d f(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i})$$

Here the subscript h means the set of all homogeneous polynomials and d is the degree of f.

(a) Show that $\mu_i(\nu_i(f)) = f$

(b) Show that μ_i and ν_i are not inverses