Exercises 6/12/08

1. Find all projective cubics thorough $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.

2. Let L_1 and L_2 be distinct lines in the projective plane. Show that they intersect at precisely one point.

3. Prove that any projective conic is projectively equivalent to $x^2 = 0$ (a line), $x^2+y^2 = 0$ (2 lines), or $x^2 + y^2 + z^2 = 0$ (a nondegenerate conic).

4. Let $p_1, \ldots, p_5$ be points in the projective plane, no three on a line. Prove that there is a unique conic that goes through these points.

5. Consider the map

$$\phi(x_0, x_1, x_2, x_3) = (x_0x_1, x_0x_2, x_0x_3, x_1x_2, x_1x_3, x_2x_3)$$

   (a) Does this give a well-defined map from $\mathbb{P}^2(\mathbb{Q}) \to \mathbb{P}^5(\mathbb{Q})$?

   (b) Consider the projective variety

$$X = \{(x_0 : x_1 : x_2 : x_3) | x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0\} \subseteq \mathbb{P}^3(\mathbb{C})$$

   Does the map ϕ give a well-defined map from $X \to \mathbb{P}^5(\mathbb{C})$?

Theorem 3. *(Important linear algebra fact). Let $a, b, c, d \in \mathbb{P}^2$ be points in the projective plane such that no three lie on a line. Then there exists an invertible 3×3 matrix A such that $A * a = (1, 0, 0)$, $A * b = (0, 1, 0)$, $A * c = (0, 0, 1)$ and $A * d = (1, 1, 1)$.*