

Exercises 5/28/08

1. Draw in the tangent cones on the previous set of problems. Use different colors to help distinguish the tangent cone from the graph. Also, calculate the multiplicity of each singularity.
2. Here are some harder curves. Find all singular points, the multiplicity of the singular points, and the tangent cones. Draw pictures of the curves and the tangent cones. Make the computer do as much of the work for you as possible!
  - a.  $2x^4 - 3x^2y + y^2 - 2y^3 + y^4 = 0$
  - b.  $2y^2(x^2 + y^2) - 3y^2 - x^2 + 1 = 0$
  - c.  $2y^2(x^2 + y^2) - 2y^2(x + y) - 2y^2 - x^2 + 2x + 2y = 0$
3. Show that the curve  $x^2 - 2xy - x + y + \frac{1}{4} - y^3 = 0$  is affine equivalent to our good friend  $y^2 - x^2 - x^3 = 0$ .
4. Calculate  $I((0, 0), f, l)$ , the intersection multiplicity of all lines,  $l$ , through the origin for the “eight curve”  $f(x, y) = y^2 - x^2 + x^4$ .
5. Show that multiplicity is invariant under affine equivalence. That is, if  $\phi : C_1 \rightarrow C_2$  is an affine equivalence between two curves, it maps a point with multiplicity  $m$  to a point with multiplicity  $m$ .
6. Prove that intersection numbers are invariant under affine equivalence.
7. Let  $g \in \mathbb{C}[t]$  be a polynomial with  $g(0) = 0$ .
  - (a) Prove  $t = 0$  is a root of multiplicity  $\geq 2$  of  $g$  if and only if  $g'(0) = 0$ .
  - (b) More generally, prove that  $t = 0$  is a root of multiplicity  $\geq k$  if and only if  $g'(0) = g''(0) = \dots = g^{(k-1)}(0) = 0$ .
8. We saw the definition of tangent line at a nonsingular point is the unique line with intersection multiplicity  $\geq 2$ . If  $(a, b) \in V(f)$  and  $\nabla f(a, b) \neq (0, 0)$ , prove that the tangent line of  $V(f)$  at  $(a, b)$  is defined by the equation
 
$$\partial f / \partial x(a, b)(x - a) + \partial f / \partial y(a, b)(y - b) = 0.$$
9. The four-leaved rose is defined in polar coordinates by the equation  $r = \sin(2\theta)$ . In cartesian coordinates, this curve is defined by the equation  $(x^2 + y^2)^3 = 4x^2y^2$ . Using a computer draw this curve.
  - (a) Show that most lines through the origin meet the rose with multiplicity 4 at the origin. Give a geometric explanation for this number.
  - (b) Find the lines through the origin that meet the rose with multiplicity  $> 4$ . Give a geometric explanation for the numbers you get.