Exercises 5/29/08

1. Prove properties 7 and 8 using properties 1-6.

2. How many times do the two given curves intersect at the origin?

   (a) $y = x^3$ and $y^4 + 6x^3y + x^8 = 0$

   (b) $y = x^2 - 2x$ and $y^2 + 5y = 4x^3$

   (c) $y^2 + x^2y - x^3 = 0$ and $y^2 + x^3y + 2x = 0$

   (d) $y^5 = x^7$ and $y^2 = x^3$

   (e) $xy^4 + y^3 = x^2$ and $y^5 + x^2 = xy$

3. Let C and D be two circles through the origin, and assume that the center of C lies on the x-axis. Prove that C and D intersect either once or twice at the origin, depending on whether or not the center of D lives on the x-axis.

4. Prove that if $\deg(f) = m$ and $\deg(g) = n$, then $I_{\mathbf{O}}(f, g) \leq mn$.

5. Show that the graph of the equation $r = \sin(3\theta)$ in polar coordinates corresponds to a curve $f(x, y) = 0$ of degree 4. Prove that there exist 3 lines through the origin that intersect the curve more than 3 times there and that all other lines through the origin intersect the curve exactly 3 times there. Draw the curve and the 3 exceptional lines.

6. Prove that a plane curve of degree n with a point of multiplicity $n - 1$ is rational.

7. Let C be a curve and P a point on the curve. Then P is called a *flex* or *inflection point* of C if there exists a line L through P with intersection multiplicity ≥ 3. Find a necessary and sufficient condition for there to exist a line L and a point P so that P is a flex point with respect to this line on the curve C.

8. Show that affine equivalence preserves rationality. That is, show that if $f(x, y) = g(\phi(x, y))$ for some affine equivalence ϕ and $g(x, y) = 0$ is rational then $f(x, y) = 0$ is also rational.

9. (a) Curves of degree 2 are called conics. Show that any conic $f(x, y) = ax^2 + bxy + cy^2 + dx + ey + f = 0$ is affine equivalent to one of the form $g(x, y) = ax + by + cx^2 + dxy + ey^2$.

   (b) Using #6 and #8 conclude that any conic is rational.