Exercises 5/30/08

1. Prove that the following curves are rational by finding a rational parametrization for them.

   (a) $f(x, y) = y^2 - x^3$

   (b) $f(x, y) = x^2 - y^2 - (x - 2y)(x^2 + y^2)$

   (c) $f(x, y) = x^2 + 2xy + y^2 - y$

   (d) $f(x, y) = x^2 - 2x - y + 1$

2. Consider the lemniscate $(x^2 + y^2)^2 = x^2 - y^2$. Find the parametric form of the curve, i.e., $x(t), y(t)$.

3. Use a computer algebra system to compute the following gcd's.

   (a) $\gcd(x^4 + x^2 + 1,\ x^4 - x^2 - 2x - 1,\ x^3 - 1)$

   (b) $\gcd(x^3 + 2x^2 - x - 2,\ x^3 - 2x^2 - x + 2,\ x^3 - x^2 - 4x + 4)$

4. Decide whether $x^2 - 4$ is in $< x^3 + x^2 - 4x - 4, x^3 - x^2 - 4x + 4, x^3 - 2x^2 - x + 2 >$. If so, can you express it as a combination of the generators of the ideal?

5. Let $I \subset \mathbb{C}[x, y]$ be an ideal, and let $f_1, \ldots, f_s \in \mathbb{C}[x, y]$. Prove that the following statements are equivalent:

   (a) $f_1, \ldots, f_s \in I$.

   (b) $< f_1, \ldots, f_x > \subset I$.

6. Use the previous exercise to prove the following equalities of ideals in $\mathbb{C}[x, y]$:

   (a) $< x + y,\ x - y >=< x, y >$

   (b) $< x + xy,\ y + xy,\ x^2,\ y^2 >=< x, y >$

   (c) $< 2x^2 + 3y^2 - 11,\ x^2 - y^2 - 3 >=< x^2 - 4,\ y^2 - 1 >$