

Exercises 6/2/08 (Mostly taken from [CLO] chapter 2, sections 6 & 7)

1. Compute  $S(f, g)$  using lex order:
  - (a)  $f = 4x^2z - 7y^2$ ,  $g = xyz^2 + 3xz^4$ .
  - (b)  $x^4y - z^2$ ,  $g = 3xz^2 - y$ .
  - (c)  $f = x^7y^2z + 2ixyz$ ,  $g = 2x^7y^2z + 4$ .
  - (d)  $g = xy + z^3$ ,  $g = z^2 - 3z$ .
2. Determine whether the following sets  $G$  are Gröbner basis for the ideal they generate. Use a computer algebra system to compute the  $S$ -polynomials and remainders.
  - (a)  $G = \{x^2 - y, x^3 - z\}$  grlex order
  - (b)  $G = \{xy^2 - xz + y, xy - z^2, x = yz^4\}$  lex order
3. Use Buchberger's algorithm to find a Groebner basis for each of the following ideals. (Use computer algebra to do the steps, not the whole thing!) Use the lex then the grlex order in each case, and then compare your results.
  - (a)  $I = \langle x^2y - 1, xy^2 - x \rangle$ .
  - (b)  $I = \langle x^2 + y, x^4 + 2x^2y + y^2 + 3 \rangle$ .
  - (c)  $I = \langle x - z^4, y - z^5 \rangle$ .
4. A **reduced Gröbner basis** for a polynomial ideal  $I$  is a Gröbner basis  $G$  for  $I$  such that (i)  $LC(p) = 1$  for all  $p \in G$  and (ii) For all  $p \in G$ , no monomial of  $p$  lies in  $\langle LT(G - \{p\}) \rangle$ . (Notation: If  $F = \{f_1, \dots, f_s\}$  is a basis for an ideal, then  $\langle LT(F) \rangle = \langle LT(f_1), \dots, LT(f_s) \rangle$ ). An algorithm for computing a reduced Gröbner basis for  $I$  is given in [CLO] pg 90.

Compute reduced Gröbner bases for each ideal in problem 3.
5. By Macaulay's theorem (**Theorem** (Macaulay)). Let  $I$  be an ideal of  $\mathbb{C}[x, y]$ . For any monomial order  $>$  on  $S$ , the set of all monomials not in  $LT(I)$  forms a basis for  $\mathbb{C}[x, y]/I$ . We can use Gröbner bases to compute the dimension of  $\mathbb{C}[x, y]/I$ . Using Gröbner bases and the definition of intersection multiplicity given by  $I_{\mathbf{O}}(f, g) = \dim_{\mathbb{C}}(\mathbb{C}[x, y]/\langle f, g \rangle)$ , confirm some of our intersection multiplicities that we computed last week.