Exercises 6/4/08

1. Compute the resultant of $f(x) = x^5 - 3x^4 - 2x^3 + 3x^2 + 7x + 6$ and $g = x^4 + x^2 + 1$. Do these polynomials have a common factor in $\mathbb{Q}[x]$?

2. Let $f, g \in \mathbb{C}[x]$ be polynomials of degree 3. Prove that the following are equivalent:
   (a) f and g have a common nonconstant factor.
   (b) $\mathrm{Res}(f, g, x) = 0$.

3. Resultants can give us another method of implicitization, in the following way: Given a rational curve defined parametrically by $x = \frac{f_1(t)}{g_1(t)}$ and $y = \frac{f_2(t)}{g_2(t)}$, we can find an implicit equation for the curve by calculating the resultant of the polynomials $f = f_1(t) - xg_1(t)$ and $g = f_2(t) - yg_2(t)$, where f and g are polynomials in t with coefficients in $\mathbb{C}[x, y]$. Use this method to find an implicit equation for the following curves:
   (a) $x = t^2$, $y = t^2(t + 1)$
   (b) $x = \frac{t-1}{t^2}$, $y = t - 1$
   (c) $x = \frac{t(t^2+1)}{t^4+1}$, $y = \frac{t(t^2-1)}{t^4+1}$

4. Use the Gröbner basis method to find implicit equations for the parametric curves in the previous problem, and check that they define the same curves.

5. Prove the following statement:
   If $I = \langle f, g \rangle \subset k[x, a_0, \ldots, a_m, b_0, \ldots, b_n]$ is the ideal generated by
   $$f = a_m x^m + a_{m-1}x^{m-1} = \ldots a_0, \qquad g = b_n x^n + b_{n-1}x^{n-1} + \ldots + b_0$$
   then $\mathrm{Res}(f, g, x) \in I_1 = I \cap k[a_0, \ldots, a_m, b_0, \ldots, b_n]$.

6. Use a computer algebra system to calculate $\mathrm{Res}(f, g, x)$ with
   $$f = a_2x^2 + a_1x + a_0, \qquad g = b_3x^3 + b_2x^2 + b_1x + b_0$$

7. Consider the following polynomials in $k[x, y]$ (k any field):
   $$f = x^2y - 3xy^2 + x^2 - 3xy$$
   $$g = x^3y + x^3 - 4y^2 - 3y + 1$$
   (a) Compute $\mathrm{Res}(f, g, x)$.
   (b) Compute $\mathrm{Res}(f, g, y)$.
   (c) What do your answers from (a) and (b) say about f and g?

Suggestions for further study: [CLO, chapter 3.5, 3.6] & Prof. Hassett's book on Algebraic Geometry. It would be especially nice if someone wants to explain the Extension Theorem in §3.6 of [CLO] and what that has to do with the resultant. Some nice problems that have to do with implicitization and the extension theorem are 8 and 9 of [CLO, pg. 132].