Exercises 6/5/08

1. Prove that f has a multiple factor (that is, f is divisible by g^2 for some $g \in k[x]$ of positive degree) if and only if $\text{disc}(f) = 0$.

2. If $f, g \in \mathbb{C}[x]$ are polynomials of positive degree, prove that f and g have a common root in $\mathbb{C}$ if and only if $\text{Res}(f, g, x) = 0$.

3. Let

$$\begin{aligned} f(x) &= a_m x^m + a_{m-1} x^{m-1} + \ldots + a_0 \\ g(x) &= b_n x^n + b_{n-1} x^{n-1} + \ldots + b_0 \end{aligned}$$

with $a_m \neq 0$, $b_n \neq 0$. Suppose that the $(m+n) \times (m+n)$ Sylvester matrix has rank $m + n - 1$. Show that f and g share a common linear factor, but not a common quadratic factor.

4. Let $f, g \in \mathbb{C}[x, y]$. In this exercise, you will prove that

$$V(f, g) \text{ is infinite} \iff f \text{ and } g \text{ have a nonconstant common factor in } \mathbb{C}[x, y]$$

   (a) Prove that $V(f)$ is infinite when f is nonconstant. Hint: Suppose f has positive degree in x. Then the LC of x in f can vanish for at most finitely many values of y. Now use the fact that $\mathbb{C}$ is algebraically closed.

   (b) If f and g have a nonconstant common factor $h \in \mathbb{C}[x, y]$, then use part (a) to show that $V(f, g)$ is infinite.

   (c) If f and g have no nonconstant common factor, show that $\text{Res}(f, g, x)$ and $\text{Res}(f, g, y)$ are nonzero and conclude that $V(f, g)$ is finite.

5. The **extension theorem for two polynomials**: ([CLO, pg. 160]) Let $I = \langle f, g \rangle \subset \mathbb{C}[x_1, \ldots, x_n]$ and let I_1 be the first elimination ideal of I. Also, let $a_0, b_0 \in \mathbb{C}[x_2, \ldots, x_n]$ be the coefficients given by

$$\begin{aligned} f(x) &= a_0 x_1^l + a_1 x_1^{l-1} + \ldots + a_l \\ g(x) &= b_o x_1^m + b_1 x_1^{n-1} + \ldots + b_m. \end{aligned}$$

Suppose we have a partial solution $(c_2, \ldots, c_n) \in V(I_1)$. If $(c_2, \ldots, c_n) \notin V(a_0, b_0)$, then there exists $c_1 \in \mathbb{C}$ such that $(c_1, c_2, \ldots, c_n) \in V(I)$. Vague question: What is this saying geometrically?

6. (a) Let R be a ring. Show that $1 - x$ is invertible in $R[[x]]$, with inverse $1 + x + x^2 + \ldots$.

   (b) More generally, show that $f = c_0 + c_1 x + c_2 x^2 + \ldots \in R[[x]]$ has a multiplicative inverse in $R[[x]]$ if and only if $c_0 \neq 0$.

7. Show that $f(x,y) = x^2 - y^3$ and $g(u,v) = u^2 + u^4 - v^3$ are analytically equivalent, but neither are equivalent to $h(s,t) = st$.

8. Let $f \in \mathbb{R}[x_1, x_2]$ be a real plane curve with singularity at $(0,0)$. Prove that there exist $\lambda_1, \lambda_2 \in \mathbb{R}$ and an affine change of coordinates α given by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

so that $f(\alpha(y_1, y_2)) = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \text{h.o.t.}$ Your proof should extend easily to multiple variables. Hint: use your knowledge of linear algebra and the formula for the Taylor series expansion about $(0,0)$.

9. Suggestions for further study: Problems 8 -11 in CLO §3.6. These questions address what can go wrong when computing resultants in several variables.
Resource for analytic equivalence: Professor Hassett's primer on plane curve singularities. Book of Brieskorn and Knörrer