

Exercises 6/6/08

1. Let $f(x) = c_m + x^m + c_{m+1}x^{m+1} + \dots \in \mathbb{C}[[x]]$ with $c_m \neq 0$. Show that f is analytically equivalent to $g(z) = z^m$ at the origin.
 Hint: Rewrite f as $f = x^m + c_{m+1}x^{m+1} + \dots$ using some analytic function. Consider a partial sum $x^m + c_{m+1}x^{m+1}$ and make the substitution $x = z + \alpha z^2$. Choose α to cancel the $(m+1)$ -degree term. Then show that this can be repeated to cancel higher-degree terms.
2. Show that $f(x_1, \dots, x_n) = x_1^2 + \dots + x_n^2 + f_3(x_1, \dots, x_n) + \text{h.o.t.} \in \mathbb{C}[[x_1, \dots, x_n]]$ (that is, a formal power series whose second order homogeneous term is $f_2 = x_1^2 + \dots + x_n^2$) is analytically equivalent at the origin to $g(z_1, \dots, z_n) = z_1^2 + \dots + z_n^2$.
3. Let $f = f_m + f_{m+1} + \dots$ and $g = g_{m'} + g_{m'+1} + \dots \in \mathbb{C}[[x_1, \dots, x_n]]$. Assume f and g are analytically equivalent via $x_i = \varphi_i(y_1, \dots, y_n)$.
 - (a) Show $m = m'$
 - (b) Show $g_m = f_m \left(A \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \right)$ where

$$A = \left(\frac{\partial \varphi_i}{\partial x_j} \Big|_{(0, \dots, 0)} \right)$$
4. The plane curves $y^2 = x^2(x+1)$ and $xy = 0$ both have a singularity at the origin. Show that these singularities are analytically isomorphic by exhibiting an explicit formal power series change of variables.
5. Determine the number of branches at the origin for the following curves. Include pictures.
 - (a) The cissoid of Diocles: $y^2(1-x) - x^3 = 0$
 - (b) The conchoid of Nichomedes: $(x^2 + y^2)(x-1)^2 - x^2 = 0$
 - (c) The cardioid: $(x^2 + y^2 + 4y)^2 - 16(x^2 + y^2) = 0$
 - (d) The four-leaf rose: $(x^2 + y^2)^3 - 4x^2y^2 = 0$
6. Show that the following are inequivalent
 - (a) $f(s, t) = x^3 - t^3$
 - (b) $g(x, y) = x^3 + y^4$
 - (c) $h(u, v) = u^2v + u^4 + v^4$