

**1 Vector Spaces**

- vector space, subspace, linear combination, span, linear dependence, basis, dimension
- Subsets of linearly independent sets are linearly independent.
- Replacement Theorem
- If  $V$  is  $n$ -dimensional, any linearly independent set containing  $n$  elements is a basis.
- The dimension of a subspace is always less than or equal to the dimension of the space.

**2 Linear Transformations and Matrices**

- linear transformation, null space, range, rank, nullity, ordered basis, coordinate vector, matrix representations of linear transformations, left multiplication transformation, invertibility, inverse, change of coordinates matrix,
- $T : V \rightarrow W$  linear,  $\beta = \{v_1, \dots, v_n\}$  a basis for  $V$ , then  $R(T) = \text{span}(\{T(v_1), \dots, T(v_n)\})$ .
- $T : V \rightarrow W$  linear,  $\dim(V) = \text{nullity}(T) + \text{rank}(T)$ .
- $T : V \rightarrow W$  linear,  $\dim(V) = \dim(W)$ ,  $T$  is one-to-one if and only if  $T$  is onto.
- Properties of the matrix representations with respect to ordered bases.
- $T : V \rightarrow W$  linear. If  $T$  is invertible, then  $\dim(V) = \dim(W)$ .
- For finite dimensional vector spaces  $V, W$  over the same base field,  $V$  and  $W$  are isomorphic if and only if  $\dim(V) = \dim(W)$ .

**3 Elementary Matrix Operations and Systems of Linear Equations**

- three types of elementary operations, elementary matrices, rank of a matrix, inverse of a matrix, Gaussian Elimination, reduced row echelon form,
- Elementary row and column operations are rank preserving.
- The rank of a matrix equals the number of linearly independent columns (rows).
- properties of rank
- Every invertible matrix is the product of elementary matrices.
- Let  $Ax = 0$  be a system of  $m$  linear equations and  $n$  unknowns. If  $m < n$ , the system has a nonzero solution.
- Let  $Ax = b$  be a system of  $n$  linear equations and  $n$  unknowns.  $A$  is invertible if and only if the system has exactly one solution.
- The system  $Ax = b$  has at least one solution if and only if  $\text{rank}(A) = \text{rank}(A | b)$ .

**4 Determinants**

- definition of an  $n \times n$  determinant
- $\det(AB) = \det(A) \det(B)$ .
- Let  $E$  be an elementary matrix.

$$\det(E) = \begin{cases} -1 & \text{if } E \text{ is of type 1;} \\ k & \text{if } E \text{ is of type 2 with one row of } I \text{ multiplied by } k; \\ 1 & \text{if } E \text{ is of type 3.} \end{cases}$$

- $\det(A) = \det(A^t)$
- $A$  is invertible if and only if  $\det(A) \neq 0$ .
- The determinant of an upper triangular matrix is the product of the diagonal entries.
- If  $A$  is invertible, then  $\det(A^{-1}) = \frac{1}{\det(A)}$ .

## 5 Diagonalization

- diagonalizable, similar matrices, eigenvalue, eigenvector, eigenspace, split polynomial, characteristic polynomial, multiplicity, invariant subspaces
- $T \in \mathcal{L}(V)$  is diagonalizable if and only if there exists an ordered basis for  $V$  consisting of eigenvectors of  $T$ .
- The characteristic polynomial of an  $n \times n$  matrix is a polynomial of degree  $n$  with leading coefficient  $(-1)^n$ .
- Test for the diagonalizability of  $T \in \mathcal{L}(V)$ :
  1. The characteristic polynomial splits over the base field.
  2. For each eigenvalue  $\lambda$ , the multiplicity of  $\lambda$  equals  $\dim(E_\lambda) = n - \text{rank}(T - \lambda 1)$ .
- Cayley-Hamilton Theorem

## 6 Inner Product Spaces

- inner product, adjoint of a matrix, norm, inner product space, orthogonal, unit vector, orthonormal, orthonormal basis, orthogonal complement, orthogonal projection, adjoint of a transformation, normal transformation and matrix, self-adjoint, unitary operator (matrix), orthogonal operator(matrix), unitarily (orthogonally) equivalent
- Gram-Schmidt Orthogonalization Process
- A set  $S = \{v_1, v_2, \dots\}$  is orthonormal if and only if  $\langle v_i, v_j \rangle = \delta_{ij}$ .
- An orthogonal set of nonzero vectors is linearly independent.
- If  $W \subseteq V$ , then  $\dim(V) = \dim(W) + \dim(W^\perp)$ .
- Schur's Theorem
- Equivalences to being unitary or orthogonal.
- Every eigenvalue of a self-adjoint operator is real.
- Let  $T \in \mathcal{L}(V)$  where  $V$  is a finite dimensional real inner product space. Then  $T$  is both self-adjoint and symmetric if and only if there exists an orthonormal basis for  $V$  of eigenvectors of  $T$  corresponding to eigenvalue of absolute value 1.
- Let  $T \in \mathcal{L}(V)$  where  $V$  is a finite dimensional complex inner product space. Then  $T$  is unitary if and only if there exists an orthonormal basis for  $V$  of eigenvectors of  $T$  corresponding to eigenvalue of absolute value 1.
- Let  $A \in M_{n \times n}(\mathbb{C})$ .  $A$  is normal if and only if  $A$  is unitarily equivalent to a diagonal matrix.
- Let  $A \in M_{n \times n}(\mathbb{R})$ .  $A$  is symmetric if and only if  $A$  is orthogonally equivalent to a diagonal matrix.
- $|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$  (Cauchy-Schwartz Inequality)
- $\|x + y\| \leq \|x\| + \|y\|$  (Triangle Inequality)

## 7 Jordan Canonical Form

- Jordan block, Jordan canonical form, Jordan canonical basis, generalized eigenvectors and eigenspaces, cycle of generalized eigenvectors, dot diagrams
- For a fixed eigenvalue  $\lambda$  of  $T \in \mathcal{L}(V)$ ,  $\dim(K_\lambda(T))$  equals the multiplicity of  $\lambda$ .
- The number of Jordan blocks corresponding to an eigenvalue  $\lambda$  equals  $\dim(E_\lambda)$ .
- Let  $T$  be a linear transformation on a finite dimensional vector such that the characteristic polynomial of  $T$  splits. Then  $T$  is diagonalizable if and only if  $K_\lambda(T) = E_\lambda$  for each eigenvalue  $\lambda$  of  $T$ .
- If  $A$  is an  $n \times n$  matrix whose characteristic polynomial splits, then  $A$  has a Jordan canonical form  $J$ , and  $A$  is similar to  $J$ .
- Two matrices are similar if and only if they have the same Jordan canonical form (up to the ordering of the eigenvalues).