Math 499 Exercises
February 2, 2005

1. Prove that the curves $C(f)$ are rational by finding rational functions $x(t), y(t)$, not both constant, such that $f(x(t), y(t)) \equiv 0$.
   Hint: I usually start by trying $t = \frac{y}{x}$, but this will not work every time.

   (a) $f = y^2 - x^3$

   (b) $f = x^2 - y^2 - (x - 2y)(x^2 + y^2)$

   (c) $f = x^5 - xy^2 + y^3$

   (d) $f = 3x - 2y - y^2$

   (e) $f = x^5 - x^4 + x^2y - y^2$

   (f) $f = x^2 + 2xy + y^2 - y$

   (g) $f = x^2 - 2x - y + 1$

   Example: $f = y^2 - x^2 - x^3$
   I start by assuming $x \neq 0$ and factoring out my highest power of x to get

   $$x^3(\frac{y^2}{x^3} - \frac{1}{x} - 1) = 0.$$

   If we let $t = y/x$ this is just

   $$x^3(t^2\frac{1}{x} - \frac{1}{x} - 1) = 0.$$

   We already assumed that $x \neq 0$ so we must have that

   $$t^2\frac{1}{x} - \frac{1}{x} = 1.$$

   We can multiply both sides by x to see that $x(t) = t^2 - 1$ and $y(t) = t(t^2 - 1)$. Now we need to check that $f(x(t), y(t)) \equiv 0$.

   $$\begin{aligned} f\left(t^2 - 1, t(t^2 - 1)\right) &= t^2(t^2 - 1)^2 - (t^2 - 1)^2 - (t^2 - 1)^3 \\ &= t^2(t^2 - 1)^2 - (t^2 - 1)^2 - (t^2 - 1)(t^2 - 1)^2 \\ &= (t^2 - 1)^2[t^2 - 1 - (t^2 - 1)] \\ &= 0 \end{aligned}$$

2. Prove the following:

   (a) Any irreducible conic, C(f) where degree of f is 2, is rational.
   (We can choose our coordinates x, y such that the curve passes through the origin and so an irreducible conic can be written $f(x, y) = f_1(x, y) + f_2(x, y) = (c_1x + c_2y) + (c_3x^2 + c_4xy + c_5y^2)$ where $c_i \in \mathbb{C}$.)

   (b) Let f_i denote the degree i part of the polynomial f. So after choosing our coordinates so that $f(0, 0) = 0$, we can write any polynomial as $f = f_1 + f_2 + \ldots$ Prove that an irreducible curve of degree n, $C(f)$, is rational if $f = f_{n-1} + f_n$. (This is a generalization of the previous problem.)