Math 499 Exercises
February 23, 2005

1. Let $f(t), g(t)$ be polynomials in $\mathbb{C}(t)$ of degrees m, n, respectively. In class I stated that they have a common zero if and only if they have a common nonconstant factor $h(t) \in \mathbb{C}(t)$ which happens if and only if $Res(f, g) = 0$.

   In the case that f, g are both of degree 3 prove the following are equivalent:

   (i) f and g have a common nonconstant factor.

   (ii) Res(f, g)=0.

   (Hint: A linear algebra book may come in handy! I particulary enjoy reading up on null space and properties of determinants.)

   Find the pattern in the case of $m = n = 3$ that lets you prove this result for arbitrary $m, n \in \mathbb{N}$.

2. When you are given the parametric form of a curve $x = f_1(t)/g_1(t)$ and $y = f_2(t)/g_2(t)$, you can find the implicit equation by computing the resultant of the polynomials $f = f_1(t) - xg_1(t)$ and $g = f_2(t) - yg_2(t)$ where you consider your polynomials as polynomials with variable t and coefficients in $\mathbb{C}[x, y]$.
   Use the method of resultants to find the implicit form of the following curves.

   (a) $x = t^2, \;\; y = t^2(t + 1)$

   (b) $x = \frac{t-1}{t^2}, \;\; y = t - 1$

   (c) $x = \frac{t(t^2+1)}{1+t^4}, \;\; y = \frac{t(t^2-1)}{t^4+1}$

3. Use Groebner Basis and Maple to find the implicit equation of the parametric curves above and check that they define the same curves.

4. Use the fact that $f(t)$ has a multiple root at $\alpha \in \mathbb{C}$ if and only if $f(\alpha) = f'(\alpha) = 0$ to decide whether these polynomials have distinct roots.

   (a) $x^3 + 6x^2 + 12x + 8.$

   (b) $x^4 + 10x^3 + 27x^2 - 54$

   (c) $x^3 + 3x + 2$

For resultant help check out: http://www.math.rice.edu/ hassett/CAGbook/CAGch6.pdf