Math 499 Exercises
January 19, 2005

1. Find the singular points of these curves by taking partial derivatives. Next, find the Taylor expansion of the curve about the singular points you found. If the lowest degree terms in the Taylor expansion are of degree 2, then determine if the point is a node or a cusp. (Remember: $\Delta = \frac{\partial^2 f}{\partial x \partial y} | p - \left(\frac{\partial^2 f}{\partial x \partial y} | p\right)^2$ and $p$ is a cusp if $\Delta = 0$ and a node otherwise.) Finally, use Maple to plot the curves using "implicitplot" while avoiding the singularities.

(a) $x^2 = x^4 + y^4$

(b) $xy = x^6 + y^6$

(c) $x^3 = y^2 + x^4 + y^4$

(d) $x^2y + xy^2 = x^4 + y^4$

(e) $y^2 = x^2 + x^3$

(f) $(x^2 + y^2)^2 = x^2 - y^2$

2. Prove the following:

(a) $y^2 = x^3 + px + q$ has no singular points if and only if $x^3 + px + q$ has three distinct zeros. Hint: For polynomial in one variable, $f(x)$ has a multiple root at a point $a$ if and only if $f(a) = f'(a) = 0$.

(b) For any two distinct points $P_1, P_2$ of an irreducible curve, there exists a rational function $u$ such that $u(P_1) = 0$ and $u(P_2) = 1$. Hint: You can assume that $P_1$ is $(0,0)$.