

A primer on plane curve singularities

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This note is a quick orientation to the theory of plane curve singularities. We will not give proofs or even completely precise definitions in all cases. Rather, this is a guidebook to the existing references on the subject.

1 Plane curves

By definition, a *complex plane curve* is the locus of points in the complex plane where a nonconstant polynomial vanishes. Symbolically, given $f \in \mathbb{C}[x, y]$ we have

$$C(f) := \{(x, y) \in \mathbb{C}^2 : f(x, y) = 0\} \subset \mathbb{C}^2.$$

For example, there is the Fermat curve

$$x^3 + y^3 = 1$$

and the complex solutions to the equation for the unit circle

$$x^2 + y^2 = 1.$$

A result from abstract algebra says that every $f \in \mathbb{C}[x, y]$ can be expressed uniquely as a product of powers of distinct irreducible polynomial factors

$$f = f_1^{e_1} \dots f_r^{e_r}.$$

An *irreducible plane curve* is one determined by an irreducible polynomial. In general, we have

$$C(f) = C(f_1 f_2 \dots f_r) = C(f_1) \cup \dots \cup C(f_r),$$

i.e., every plane curve is a union of irreducible plane curves and can be determined by a polynomial without multiple factors. In what follows, we'll always assume f has no multiple factors.

2 Singularities

A point $p \in C(f)$ is *singular* if the partial derivatives of f both vanish at p ; a plane curve is *nonsingular* or *smooth* if it has no singular points. The *singular locus* $\text{Sing}(C(f)) \subset C(f)$ is thus determined by the equations

$$f(x, y) = 0 \quad \frac{\partial f}{\partial x}(x, y) = \frac{\partial f}{\partial y}(x, y) = 0,$$

which guarantee that $(x, y) \in C(f)$ and the gradient vector of f vanishes at (x, y) .

The condition of nonsingularity allows us to apply the implicit function theorem from calculus to express one of the variables in terms of the other along $C(f)$: If $\partial f / \partial y|_p \neq 0$ then y can be expressed as a complex differentiable function of x in a neighborhood of p in $C(f)$.

For example, if we take $f = y^2 - x^2 - x^3$ we have the system

$$y^2 - x^2 - x^3 = 0 \quad -2x - 3x^2 = 2y = 0$$

and the singular locus is the origin. When $y \neq 0$ (and hence $x \neq 0, 1$) we can express

$$y = \sqrt{x^2 + x^3}$$

for a suitable branch of the complex square-root function.

It is generally true that the singular locus of a plane curve is finite, provided f has no multiple factors. In particular, about each singular point $p \in C(f)$ there is an open neighborhood $U \ni p, U \subset C(f)$, so that p is the only singular point of $C(f)$ over U . Thus plane curve singularities are *isolated*.

If $(a, b) \in C(f)$ then the Taylor expansion for f about (a, b)

$$f = f_{1,0}(x - a) + f_{0,1}(y - b) + f_{2,0}(x - a)^2 + \text{higher order terms}$$

has no constant term. Furthermore, $(a, b) \in C(f)$ is singular if and only if

$$f_{1,0} = \frac{\partial f}{\partial x}|_{(a,b)} = 0 \quad f_{0,1} = \frac{\partial f}{\partial y}|_{(a,b)} = 0,$$

i.e., the Taylor expansion has no terms of order ≤ 1 .

The Taylor expansion of the polynomial determining a plane curve singularity encodes many of its geometric properties. Here is a concrete example:

Again consider $C(f)$, $f = y^2 - x^2 - x^3$ about the origin. Note that f is irreducible in $\mathbb{C}[x, y]$. However, we can write

$$f = (y - x\sqrt{1+x})(y + x\sqrt{1+x})$$

and expand

$$\sqrt{1+x} = 1 + \sum_{n=1}^{\infty} \frac{(-1) \dots (1-2n)}{2^n} x^n$$

to obtain

$$f = \left(y - x - \sum_{n=1}^{\infty} \frac{(-1) \dots (1-2n)}{2^n} x^{n+1} \right) \left(y + x + \sum_{n=1}^{\infty} \frac{(-1) \dots (1-2n)}{2^n} x^{n+1} \right).$$

Thus the Taylor series for f at $(0, 0)$ is *reducible* in the ring of all such power series. Furthermore, near the origin we have isomorphisms

$$\begin{aligned} C(y^2 - x^2 - x^3)_{(0,0)} &\rightarrow C(\bar{y}^2 - \bar{x}^2)_{(0,0)} \\ (\bar{y}, \bar{x}) &\mapsto \left(y, x + \sum_{n=1}^{\infty} \frac{(-1) \dots (1-2n)}{2^n} x^{n+1} \right) \end{aligned}$$

Figures 1 and 2 may help you to visualize these local isomorphisms. As we take successively smaller neighborhoods about $(0, 0)$, the plane curves $C(y^2 - x^2 - x^3)$ and $C(y^2 - x^2)$ look increasingly similar.

3 Local analysis of singularities

A *formal power series* about the origin is an infinite sum

$$g = \sum_{i,j \geq 0} g_{i,j} x^i y^j, \quad g_{i,j} \in \mathbb{C}.$$

This is formal in the sense that we do not care about convergence of the series. Formal power series can be added and multiplied in the standard way; the resulting ring is denoted $\mathbb{C}[[x, y]]$. Of course, it contains the ring of polynomials $\mathbb{C}[x, y]$.

Here are a few facts about formal power series:

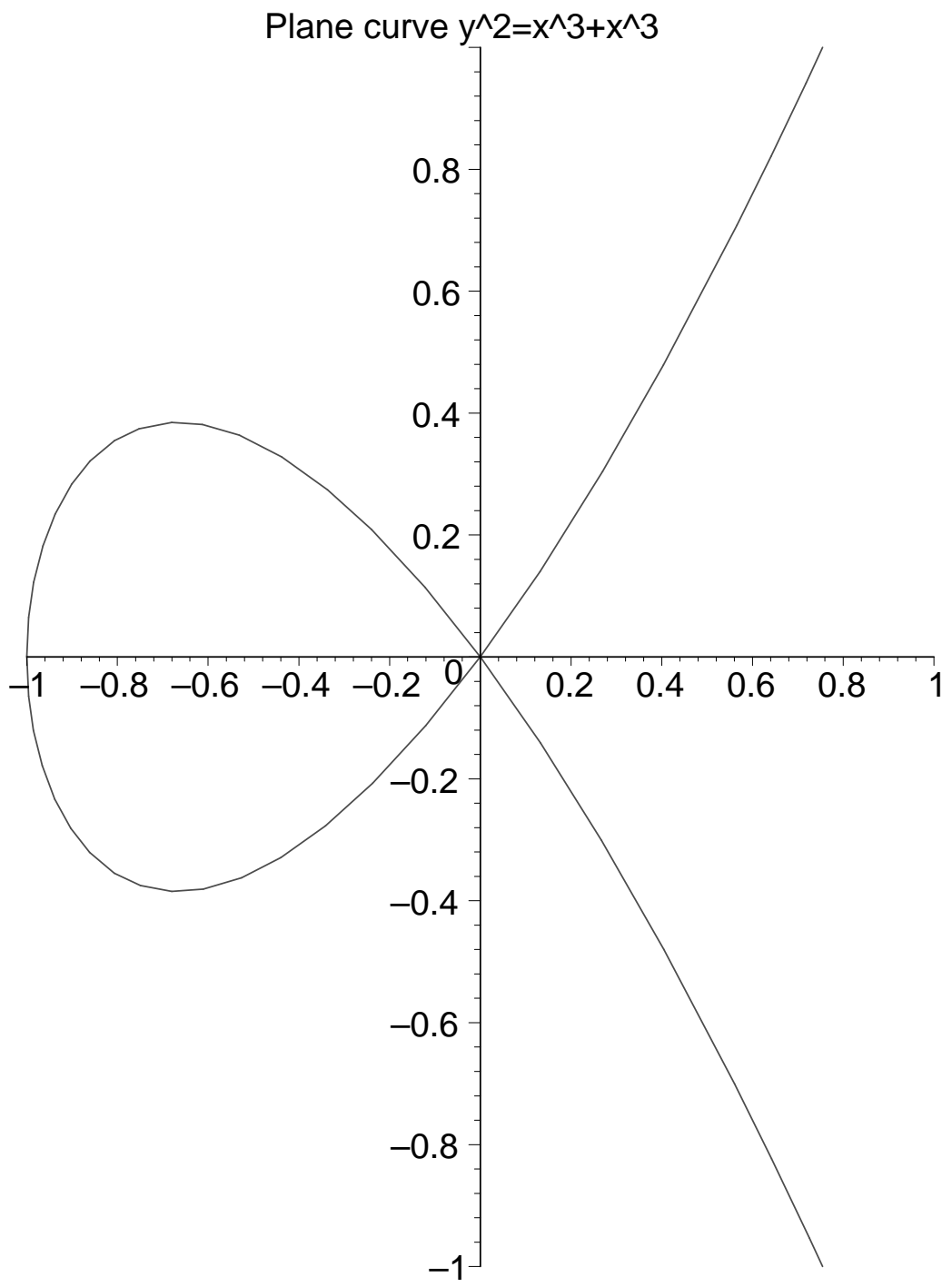


Figure 1:

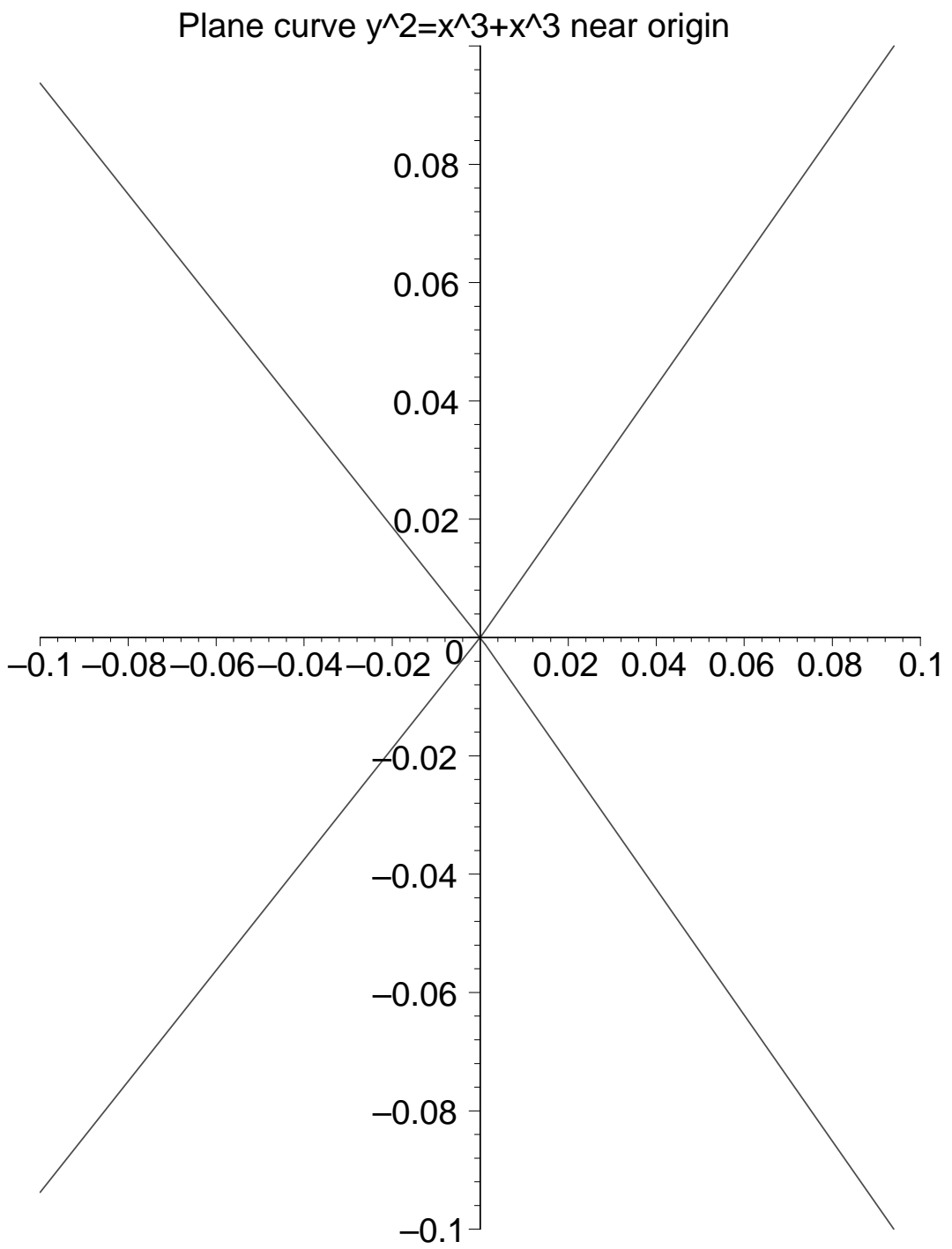


Figure 2:

1. Any formal power series with nonzero constant term admits a multiplicative inverse and has N th roots for each $N \in \mathbb{N}$, i.e., if $g \in \mathbb{C}[[x, y]]$ with $g_{0,0} \neq 0$ then $1/g$ and $\sqrt[N]{g} \in \mathbb{C}[[x, y]]$.
2. $\mathbb{C}[[x, y]]$ is a unique factorization domain.
3. A \mathbb{C} -linear homomorphism

$$\phi : \mathbb{C}[[\bar{x}, \bar{y}]] \rightarrow \mathbb{C}[[x, y]]$$

is an isomorphism if and only if the matrix

$$d\phi|_{(0,0)} = \begin{pmatrix} \frac{\partial \phi_x}{\partial \bar{x}} & \frac{\partial \phi_x}{\partial \bar{y}} \\ \frac{\partial \phi_y}{\partial \bar{x}} & \frac{\partial \phi_y}{\partial \bar{y}} \end{pmatrix}$$

is invertible.

Two formal power series $g, h \in \mathbb{C}[[x, y]]$ are *formally equivalent* if there exists an automorphism

$$\phi : \mathbb{C}[[x, y]] \rightarrow \mathbb{C}[[x, y]]$$

such that $\phi(g) = h$. A formal power series g with $g(0,0) = 0$ is *singular* if

$$g_{1,0} = \frac{\partial g}{\partial x}|_{(0,0)} = 0 \quad g_{0,1} = \frac{\partial g}{\partial y}|_{(0,0)} = 0.$$

Note that any nonsingular formal power series is formally equivalent to x .

Suppose $C(f)$ is a plane curve containing the origin. We regard f as an element of $\mathbb{C}[[x, y]]$, with a view toward extracting invariants for the singularity. We have a factorization

$$f = g_1^{e_1} \cdots g_b^{e_b}$$

where the $g_j \in \mathbb{C}[[x, y]]$ are distinct irreducible factors of f . Each $e_j = 1$ provided f has no multiple factors in $\mathbb{C}[[x, y]]$. The factors g_j are called the *branches* of $C(f)$ at $(0,0)$. Even when f is irreducible, $C(f)$ might have ≥ 2 branches at the origin. For example, $y^2 - x^2 - x^3$ is irreducible but has two branches at the origin.

Suppose now that $C(f)$ and $C(\bar{f})$ are two plane curves containing the origin. By definition, $C(f)$ and $C(\bar{f})$ are *formally isomorphic* at the origin

$$C(f)_{(0,0)} \xrightarrow{\cong} C(\bar{f})_{(0,0)}$$

if f and \bar{f} are formally equivalent in $\mathbb{C}[[x, y]]$. For example, $C(y^2 - x^2)$ and $C(y^2 - x^2 - x^3)$ are formally isomorphic at the origin. Generally, two plane curves containing (a, b) are formally isomorphic at (a, b) if

$$C(f(x + a, y + b))_{(0,0)} \xrightarrow{\cong} C(\bar{f}(x + a, y + b))_{(0,0)}.$$

Observe that $C(f)$ and $C(\bar{f})$ are guaranteed to be formally isomorphic at a point if they are both nonsingular at that point.

Here are some examples:

1. $C(y^2 - x^3 - x^n)$, $n \geq 4$ and $C(y^2 - x^3)$ are formally isomorphic at the origin.
2. $C(y^3 - x^3)$ and $C(y^3 - x^3 + x^5)$ are formally isomorphic at the origin.
3. $C(y^4 - x^4)$ and $C((y^2 - x^2)(y^2 - 2x^2))$ are *not* formally isomorphic at the origin.

4 Invariants of plane curve singularities

Formal isomorphism is an equivalence relation, so we define a *plane curve singularity* as an equivalence class of plane curves singular at the origin. We describe invariants of plane curve singularities, as represented by $g \in \mathbb{C}[x, y] \subset \mathbb{C}[[x, y]]$ without multiple factors.

The *multiplicity* of g is the degree of the smallest nonzero term in

$$g = \sum_{i,j:i+j \geq \text{mult}(g)} g_{i,j} x^i y^j.$$

The *Milnor number* is defined

$$\mu(g) = \dim_{\mathbb{C}} \frac{\mathbb{C}[[x, y]]}{\langle \partial g / \partial x, \partial g / \partial y \rangle},$$

i.e., the dimension of this ring as a complex vector space. There is another definition of this quantity in terms of the topology/geometry of the ‘Milnor fibration’

$$\begin{aligned} \pi : \mathcal{C} &\rightarrow \Delta = \{t \in \mathbb{C} : |t| < \epsilon\} \\ \{(x, y) : g(x, y) = t\} &\mapsto t. \end{aligned}$$

The *Tjurina number* is defined

$$\tau(g) = \dim_{\mathbb{C}} \frac{\mathbb{C}[[x, y]]}{\langle g, \partial g / \partial x, \partial g / \partial y \rangle},$$

so that $\tau(g) \leq \mu(g)$.

1. For $g = x^p - y^q$ we have

$$\tau(g) = \mu(g) = (p-1)(q-1),$$

corresponding to the basis elements $x^i y^j, i \leq p-2, j \leq q-2$.

2. For $g = y^3 - x^7 - x^5 y$ we have

$$\tau(g) = 11 \quad \mu(g) = 12$$

corresponding to the basis elements

$$\begin{aligned} \{1, x, x^2, x^3, x^4, y, xy, x^2 y, x^3 y, y^2, xy^2, y^3\} &\in \frac{\mathbb{C}[[x, y]]}{\langle \partial g / \partial x, \partial g / \partial y \rangle} \\ \{1, x, x^2, x^3, x^4, y, xy, x^2 y, x^3 y, y^2, xy^2\} &\in \frac{\mathbb{C}[[x, y]]}{\langle g, \partial g / \partial x, \partial g / \partial y \rangle}. \end{aligned}$$

5 Families of plane curves

The *universal family of plane curves of degree $\leq d$* is expressed

$$F := t_{d,0}x^d + t_{d-1,1}x^{d-1}y + \dots + t_{0,d}y^d + \dots + t_{1,0}x + t_{0,1}y + t_{0,0} = 0$$

where the $t_{i,j}$ are unassigned values of the parameters. For each point

$$[a_{d,0}, \dots, a_{0,0}] \in \mathbb{CP}^{\binom{d+2}{2}-1}$$

we can specialize $t_{i,j} = a_{i,j}$ in F to get the equation of a plane curve. Projection onto the projective space factor gives a morphism

$$\mathcal{U}_d := C(F) \subset \mathbb{C}^2 \times \mathbb{CP}^{\binom{d+2}{2}-1} \xrightarrow{\pi} \mathbb{CP}^{\binom{d+2}{2}-1}$$

whose fibers are plane curves of degree $\leq d$. These fibers might be empty and may have degree $< d$, e.g., when $a_{d,0} = a_{d-1,1} = \dots = a_{0,d} = 0$.

Now *universal* has a technical sense that requires some explanation. First, we need to define the notion of a *pull-back*. For each morphism from a variety

$$\nu : B \rightarrow \mathbb{CP}^{\binom{d+2}{2}-1}$$

the composed $t_{i,j} \circ \nu$ are defined on B , and we obtain a family of degree d plane curves over B

$$\begin{aligned} \pi^{-1}(\nu(B)) &\rightarrow B \\ \{t_{d,0}(\nu(b))x^d + \dots + t_{0,0}(\nu(b)) = 0\} &\mapsto b. \end{aligned}$$

This is called the *pull-back of the universal family via ν* and is denoted

$$\nu^*\mathcal{U}_d \rightarrow B.$$

The universal property then takes the following form: Given a family $\mathcal{C} \rightarrow B$ of plane curves of degree $\leq d$, there exists a morphism $\nu : \mathcal{C} \rightarrow \mathbb{CP}^{\binom{d+2}{2}-1}$ and an isomorphism $\mathcal{C} \simeq \nu^*\mathcal{U}_d$ over B .

Here is a concrete example: The universal family of plane conics is

$$\begin{aligned} \mathcal{U}_2 &\xrightarrow{\pi} \mathbb{CP}^5 \\ \{a_{2,0}x^2 + a_{1,1}xy + a_{0,2}y^2 + a_{1,0}x + a_{0,1}y + a_{0,0} = 0\} &\mapsto [a_{2,0}, a_{1,1}, a_{0,2}, a_{1,0}, a_{0,1}, a_{0,0}]. \end{aligned}$$

Consider the family of ellipses centered at the origin

$$\begin{aligned} \mathcal{C} &\rightarrow \mathbb{C}^3 \\ \{sx^2 + ty^2 = r^2\} &\mapsto (s, t, r) \end{aligned}$$

over \mathbb{C}^3 . The morphism ν is given by substituting

$$a_{2,0} = s \quad a_{1,1} = 0 \quad a_{0,2} = t \quad a_{1,0} = 0 \quad a_{0,1} = 0 \quad a_{0,0} = -r^2.$$

Making this substitution into \mathcal{U}_2 , we obtain the family $\mathcal{C} \rightarrow \mathbb{C}^3$.

There is an analogous theory for plane curve singularities. Fix a plane curve singularity at the origin represented by $g \in \mathbb{C}[[x, y]]$. An example of a *local deformation* of g is a family

$$\{G(t_1, \dots, t_N; x, y) = 0\} \quad G(t_1, \dots, t_N; x, y) \in \mathbb{C}[[t_1, \dots, t_N]][[x, y]]$$

so that $G(0, \dots, 0; x, y) = g(x, y)$. Again, we think of the t_j as unassigned parameters. Note that if $g \in \mathbb{C}[x, y]$ is a polynomial of degree d , then the universal family $\mathcal{U}_d \rightarrow \mathbb{C}\mathbb{P}^{\binom{d+2}{2}-1}$ induces a local deformation of g : Just recenter the coordinates in the universal family so that $g(x, y)$ occurs when all the parameters are zero.

A *(semi-uni)versal deformation* of a plane curve singularity is expressed

$$g(x, y) + \sum_{j=1}^{\tau(g)} t_j x^{a_j} y^{b_j} = 0$$

where the $x^{a_j} y^{b_j}$ are a monomial basis for the $\tau(g)$ -dimensional quotient ring

$$\frac{\mathbb{C}[[x, y]]}{\langle g, \partial g / \partial x, \partial g / \partial y \rangle}.$$

The $t_1, \dots, t_{\tau(g)}$ are called the *parameters of the versal deformation*.

Theorem 5.1 *Every deformation of a plane curve singularity is formally isomorphic to a pull-back of the versal deformation, i.e., the parameters of the versal deformation can be specialized to obtain any deformation.*

Here is a concrete example: For $g(x, y) = y^2 - x^2$ the versal deformation is

$$y^2 - x^2 + t = 0.$$

Consider the two-parameter deformation

$$y^2 - x^2 + s_1 x + s_2 y = 0.$$

Completing the squares

$$\bar{y} = y + s_2/2 \quad \bar{x} = x - s_1/2$$

gives an isomorphism to

$$\bar{y}^2 - \bar{x}^2 + 1/4(-s_2^2 + s_1^2) = 0.$$

This is obtained by specializing

$$t = 1/4(-s_2^2 + s_1^2)$$

in the versal deformation.

Here is why this is a useful concept: The most efficient way to prove a natural assertion about all families of plane curves of degree d is to prove the corresponding statement about $\mathcal{U}_d \rightarrow \mathbb{C}\mathbb{P}^{\binom{d+2}{2}-1}$. Similarly, one proves facts about all deformations of a given plane curve singularity by proving them for the versal deformation.

6 Equisingular classes

To be added later...

7 Remarks on references

A basic reference on plane curves and their singularities is the textbook [Fu]. The most comprehensive general reference is the book [BrK], which introduces many of the concepts discussed here, while at the same time introducing the necessary background from algebraic geometry.

The basic references for equisingularity of plane curves are [Zar] and [Wa], where the space of equisingular deformations is shown to be smooth. The relation between equisingularity of plane curves and the Milnor number is established in [Te].

For further background on ideals associated to plane curves, [DH] and §1.2 of [Lo] are good references.

A computer implementation of equisingularity algorithms is [GPS].

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