Math 368. Combinatorics and Graph Theory. **Fall 2012.**

List of problems:

Problem 5, page 9.

Let G be a graph with $\delta(G) \geq k$.

(a) Prove that G has a path of length at least k.

(b) If $k \geq 2$, prove that G has a cycle of length at least $k+1$.

Solution to (a). For $k = 0$ the statement is trivial because for any $v \in V$ the sequence v (of one term in V) forms a path of length 0.

So we assume that $k \geq 1$. Let n be the maximal possible length of a path in G, and let $v_1, v_2, \ldots, v_{n+1}$ be such a path of length n. In order to prove (a), we have to show that $n \geq k$.

If not (thus the proof is conducted by contradiction), then $n < k$. Denote $S = N(v_{n+1}) \setminus \{v_i \mid 1 \leq i \leq n\}$.

We have $|S| \geq \delta(G) - n \geq k - n \geq 1$, i.e. $S \neq \emptyset$. Pick $u \in S$. Then the sequence $v_1, v_2, \ldots, v_{n+1}, u$ forms a path of length $n+1$, a contradiction with the assumption that n is the maximal length of a path in the graph G.

Solution to (b). If $\delta(G) = 1$, G does not need to have cycles (example: P_n, $n \geq 2$).

Now assume that $k \geq 2$. As in the previous proof, let n be the maximal possible length of a path in G, and let $v_1, v_2, \ldots, v_{n+1}$ be such a path of length n. It follows from (a) that $n \geq k$. Consider

$$S = N(v_{n+1}) \setminus \{v_i \mid n-k+2 \leq i \leq n\}.$$

Clearly, $|S| \leq k - (k-1) = 1$, i.e., $S \neq \emptyset$. Pick $u \in S$. Then the walk $v_1, v_2, \ldots, v_{n+1}, u$ cannot be a path because its length is $n+1 > n$. It follows that $u = v_m$, for some $m \in [1, n-k+1]$, and then $v_m, v_{m+1}, \ldots, v_{n+1}, u$ is a path of length $n+1-m+1 = n+2-m \geq n+2-(n-k+1) = k+1$, completing the proof of (b).

Problem 2 (from Test 1).

Prove that if $\mathrm{diam}(G) = 8$ then $\omega(\overline{G}) \geq 5$.

Solution. If $\mathrm{diam}(G) = 8$, there are $u, v \in V$ such that $d(u,v) = 8$. Then there exists an u_v-path

$$u_0,\ u_1,\ \ldots,\ u_7,\ u_8 \tag{1}$$

such that $u_0 = u$ and $u_8 = v$.

Claim. For integers i, j such that $0 \leq i \leq j \leq 8$, $d(u_i, u_j) = j - i$.

Proof #1 of the Claim.
This has been proved in class for any geodesic path. (A path u_v is geodesic if its length is equal to $d(u,v)$).

Proof #2 of the Claim. (Direct proof: If you do not remember the geodesic result).
Clearly, $d(u_i, u_j) \leq j - i$ because the path (1) contains a path between u_i and u_j of length $j - i$.

Now assume to the contrary that $d(u_i, u_j) < j - i$. Then by the triangle inequality

$$\begin{aligned} d(u,v) = d(u_0,u_8) &\leq d(u_0,u_i) + d(u_i,u_j) + d(u_j,u_8) \leq i + d(u_i,u_j) + (8-j) = \\ &= 8 + i - j + d(u_i,u_j) < 8 + i - j + (j-i) = 8, \end{aligned} \tag{2}$$

a contradiction with the choice of u, v such that $d(u,v) = 8$.

It folows from (2) that the distances between any two different vertices in the set

$$W = \{u_0, u_2, u_4, u_6, u_8\}$$

is at least 2, hence every pair in the set W lies in $E(\underline{G})$. $\omega(\overline{G}) \geq 5$. Since $|W| = 5$, it follows that $\omega(\overline{G}) \geq 5$.

The students who instead of writing down explicitly (2) just pointed out that all distances in the set W exceed 2 because otherwise by the triangle inequality $d(u,v) < 8$ received full credit.
