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THE WEIL-PETERSSON HESSIAN OF LENGTH ON TEICHMÜLLER SPACE

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ABSTRACT. We present a brief but nearly self-contained proof of a formula for the Weil-Petersson Hessian of the geodesic length of a closed curve (either simple or not simple) on a hyperbolic surface. The formula is the sum of the integrals of two naturally defined positive functions over the geodesic, proving convexity of this functional over Teichmüller space (due to Wolpert (1987)). We then estimate this Hessian from below in terms of local quantities and distance along the geodesic. The formula extends to proper arcs on punctured hyperbolic surfaces, and the estimate to laminations. Wolpert's result that the Thurston metric is a multiple of the Weil-Petersson metric directly follows on taking a limit of the formula over an appropriate sequence of curves. We give further applications to upper bounds of the Hessian, especially near pinching loci, and recover through a geometric argument Wolpert's result on the convexity of length to the half-power.

1. INTRODUCTION

One of the foundations of modern Teichmüller theory is Wolpert's [Wol87] theorem that the function on Teichmüller space that records the geodesic length of a simple closed curve is convex with respect to the Weil-Petersson metric. That paper provided a lower bound for the Hessian of the length function; our purpose here is to present a brief derivation of a concise formula for the Hessian in terms of natural objects on the surface associated to the curve and the tangent vectors to Teichmüller space.

To explain this result and add some context, we fix some terminology and notation. Let S be a smooth closed surface of genus g , and let $\mathcal{T}(S)$ be the Teichmüller space of (isotopy classes of) marked hyperbolic structures on S . Let $[\gamma]$ be a free homotopy class of closed curves, not necessarily simple, a slight generalization of the setting in which Wolpert worked. Typically γ will denote the representative of $[\gamma]$ that is geodesic with respect to a given metric g , with the context making it clear whether γ is the immersed curve on the surface or an immersion from a circle into the surface.

For each hyperbolic surface (S, g) , there is a geodesic representative $\gamma = \gamma_g$ of $[\gamma]$. By the uniformization theorem, as each point in $\mathcal{T}(S)$ is represented by a unique hyperbolic metric, say g , the length of the geodesic representative γ of $[\gamma]$ defines a function $\ell = \ell_\gamma = \ell_\gamma([g])$ on $\mathcal{T}(S)$. We investigate the second derivative of that function.

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The Hessian of a function is well-defined once there is a background metric. There are many metrics on $\mathcal{T}(S)$; among the more basic is the Weil-Petersson metric. Representing the tangent space to Teichmüller space as the space of Beltrami differentials which are harmonic with respect to the hyperbolic metric g representing a point in $\mathcal{T}(S)$, the Weil-Petersson metric is the L^2 metric on that space with respect to the hyperbolic area form dA_g .

The first goal of this paper is to write a formula for the Weil-Petersson Hessian of the function.

1.1. Statement of the Formula. In this subsection, we describe our formula in the simplest case; we discuss the situation of the second derivative of the length function ℓ of a closed curve, with the derivative taken along a Weil-Petersson geodesic Γ . Even in this case, we require some notation.

Let $\Gamma = \Gamma(t)$ a Weil-Petersson geodesic arc; the class $[\gamma]$ is represented by the $\Gamma(t)$ -geodesic γ_t . The tangent vector to Teichmüller space at $\Gamma(0)$ is given by a harmonic Beltrami differential, say $\mu = \frac{\Phi}{g_0}$.

We can extend $\Gamma(0)$ -Fermi coordinates along the curve γ_0 to complex coordinates in a neighborhood of (a subarc of) γ_0 . In terms of those coordinates, the quantity $-\frac{\text{Im } \Phi}{g_0} = \text{Im } \mu$ is well-defined. Let U^Φ denote the solution to the ordinary differential equation

$$(1.1) \quad U_{yy} - U = -\frac{\text{Im } \Phi}{g_0},$$

where here the geodesic is represented by a vertical line in the Fermi coordinate patch with a parametrization given by arclength.

This is enough terminology so that we may state our main result as the

Theorem 1.1. *Along the Weil-Petersson geodesic arc $\Gamma(t)$, the second variation $\frac{d^2}{dt^2}\ell$ of the length $\ell(t) = L(\Gamma(t), [\gamma])$ is given by*

$$(1.2) \quad \begin{aligned} \frac{d^2}{dt^2}\ell(t) &= \int_{\gamma_0} -2(\Delta - 2)^{-1} \frac{|\Phi|^2}{g_0^2} ds + \int_{\gamma_0} [U_y^\Phi]^2 + [U^\Phi]^2 ds \\ &= \int_{\gamma_0} -2(\Delta - 2)^{-1} \frac{|\Phi|^2}{g_0^2} ds \\ &\quad + \frac{1}{2 \sinh(\frac{\ell}{2})} \iint_{\gamma_0 \times \gamma_0} \text{Im } \mu(p) [\cosh(d(p, q) - \frac{\ell}{2})] \text{Im } \mu(q) ds(p) ds(q). \end{aligned}$$

1.2. Applications and Extensions. The formula (1.2) lends itself to applications. Using well-known techniques for solving and estimating solutions of the relevant differential equations, we obtain results in a number of different directions.

Of course, it is immediately apparent that this Hessian is positive definite, even for curves which are not simple. The first summand is written in terms of $2(\Delta - 2)^{-1} \frac{|\Phi|^2}{g_0^2}$, a ubiquitous term in Teichmüller theory whose definition

involves the global geometry of (S, g) . In Lemma 5.1, we find a locally defined lower bound for this quantity. Thus, when combined with the second expression for the second term, we obtain an estimate defined only in terms of local quantities or distance along the geodesic.

In addition to providing a means to estimate the Hessian from below, we also obtain an improvement on the basic convexity result: from an easily derived formula for the gradient of length, we will see (Corollary 8.1) immediately that $\ell^{\frac{2}{3}}$ is convex on all of Teichmüller space. A recent theorem [Wol08] of Wolpert is that this may be improved to $\ell^{\frac{1}{2}}$ being convex on all of Teichmüller space; we provide a proof for that as well. The proof is geometric in the sense that it hinges on a comparison of two harmonic diffeomorphisms of a cylinder.

It is straightforward to extend our derivations both to laminations and to proper geodesic arcs which connect cusps on a hyperbolic cusped surface. The explicit nature of formula (1.2) also lends itself to estimates from above, leading to estimates on the Weil-Petersson connection near the pinching locus. These were also recently obtained (and announced some time ago) by Wolpert [Wol08].

Our final application is a new proof of Wolpert's [Wol86b] proof that the Thurston metric is (a multiple of) the Weil-Petersson metric. We consider a sequence $\{\gamma_n\}$ of curves whose geodesic representatives are becoming equidistributed in the unit tangent bundle $T^1(S, g)$. Thurston observed that $\lim_{n \rightarrow \infty} \text{Hess} \ell_{\gamma_n}$ would be a positive definite quadratic form on $T_{(S, g)} \mathcal{T}(S)$: by taking a limit of the right-hand side of formula (1.2), we see that this Thurston metric is a multiple of the Weil-Petersson metric. Wolpert's argument followed a more quasiconformal analytic tradition, while this derivation is more Riemannian in perspective.

1.3. Organization of the paper. We organize the paper as follows. In Part 1, we derive Theorem 1.1. Computations in Teichmüller theory often require fixing a gauge; here we find it convenient to vary hyperbolic structures (S, g_t) under the condition that the identity mapping $\text{id} : (S, g_0) \rightarrow (S, g_t)$ is harmonic. This requirement gives the prescribed curvature equation a particularly convenient form, and our derivation begins with a sketch of a useful computation from [Wol89]. The rest of the derivation is self-contained, occupying sections 2-4.

Part 2 of the paper contains the applications and extensions. In section 5, we find a lower bound for the integrand in the first term; this leads to an estimate (Corollary 5.2) for the Hessian in terms of an integral along the curve of local quantities. Section 6 is devoted to a derivation of our results in the setting of a curve which connects ends of a (complete) hyperbolic punctured surface. Here, while the length of an arc is infinite, the Hessian of a regularized version of the length is positive and finite; section 7 extends our work from curves to laminations. In section 8, we use the formulae to give some geometric estimates: we quickly prove that not only is the length

ℓ of curves convex, but so is $\ell^{\frac{2}{3}}$. We then give a longer geometric argument that $\ell^{\frac{1}{2}}$ is also convex. We give a general upper bound for the Hessian, as well as estimates for the Weil-Petersson connection near the Deligne-Mumford compactification divisor. Finally, in section 9, we take a limit of formula (1.2) in Theorem 1.1 over curves that are becoming equidistributed to recover the result that the Thurston metric is a multiple of the Weil-Petersson metric.

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Part 1. A formula for the Weil-Petersson Hessian of length

We begin with a brief background discussion of the some the theory of Teichmüller space and the Weil-Petersson metric we will need. Tangent vectors to Teichmüller space at a point (S, g) are represented by 'harmonic Beltrami differentials' of the form $\mu = \frac{\bar{\Phi}}{g}$, where g is a hyperbolic metric on S and Φ is a holomorphic quadratic differential on (S, g) . The Weil-Petersson inner product of two such tangent vectors is the L^2 inner product

$$(1.3) \quad \left\langle \frac{\bar{\Phi}}{g}, \frac{\bar{\Psi}}{g} \right\rangle = \operatorname{Re} \int_S \frac{\Phi \bar{\Psi}}{g} d\operatorname{Area}_g.$$

Much is now known about this metric: by means of an introduction to the subject, the Weil-Petersson metric is not complete [Chu76] [Wol75], it is Kähler [Ahl61], negatively curved with good expressions [Roy] [Tro86] [Wol86a] for and estimates [Hua05] [LSY04] of the curvatures, it is quasi-isometric to the pants complex [Bro03], the isometry group is exactly the (extended) mapping class group [MW02]. Of course, the stimulus for this article and an important ingredient in some of the results above is that the Weil-Petersson metric is geodesically convex [Wol87].

2. THE SECOND DERIVATIVE OF LENGTH IN SPACE AND TIME.

We are interested in computing the second variation of geodesic length of a curve along a Weil-Petersson geodesic. We imagine the setting as a fixed differentiable surface S equipped with a family of metrics g_t , and on this surface there is a family of curves γ_t . The curves γ_t are all freely homotopic and may or may not be simple. The defining equation is that the curves γ_t are g_t -geodesics; we shall shortly write that equation out in coordinates.

To begin though, we separate the overall second variation of length into a term that refers only to the second variation of the metric g_t and a term that refers only to the second variation of the curve γ_t . This separation is

quite standard for a variational functional. Write the length of γ_s in the metric g_t as $L(g_t, \gamma_s)$. Then, in this language, the geodesic equation takes the form, for all t ,

$$(2.1) \quad \frac{\partial}{\partial s} \Big|_{s=s_0} L(g_t, \gamma_s) \left[\frac{\partial}{\partial s} \gamma_s \right] = 0,$$

if γ_{s_0} is a g_t -geodesic, and $\frac{\partial}{\partial s} \gamma_s$ is an infinitesimal variation of curves through γ_{s_0} .

The second variation of length of the g_t -geodesics γ_t is given by

$$(2.2) \quad \frac{d^2}{dt^2} L(g_t, \gamma_t) = D_{11}^2 L(g_0, \gamma_0)[\dot{g}, \dot{g}] + 2D_{12}^2 L(g_0, \gamma_0)[\dot{g}][\dot{\gamma}] + D_{22}^2 L(g_0, \gamma_0)[\dot{\gamma}, \dot{\gamma}].$$

where $\dot{g} = \frac{d}{dt} g_t$ and $\dot{\gamma} = \frac{d}{dt} \gamma_t$. Of course, if γ_t is a g_t -geodesic, then we write the geodesic equation (2.1) above in this notation as

$$D_2 L(g_t, \gamma_t)[\dot{\gamma}] = 0$$

and so

$$\begin{aligned} 0 &= \frac{d}{dt} D_2 L(g_t, \gamma_t)[\dot{\gamma}] \\ &= D_1 D_2 L(g_0, \gamma_0)[\dot{g}][\dot{\gamma}] + D_2 D_2 L(g_0, \gamma_0)[\dot{\gamma}, \dot{\gamma}]. \end{aligned}$$

Thus,

$$(2.3) \quad D_{12}^2 L(g_0, \gamma_0)[\dot{g}][\dot{\gamma}] = -D_{22}^2 L(g_0, \gamma_0)[\dot{\gamma}, \dot{\gamma}].$$

Substituting (2.3) into (2.2) yields that

$$(2.4) \quad \frac{d^2}{dt^2} L(g_t, \gamma_t) = D_{11}^2 L(g_0, \gamma_0)[\dot{g}, \dot{g}] - D_{22}^2 L(g_0, \gamma_0)[\dot{\gamma}][\dot{\gamma}].$$

Some remarks on this equation (2.4) are in order. First, note that the term $D_{22}^2 L(g_0, \gamma_0)[\dot{\gamma}][\dot{\gamma}]$ is non-negative, as the surface (S, g_0) is negatively curved; indeed, this second variation term is positive unless the vector field $\dot{\gamma}$ is tangent to the curve γ_0 . Thus our task is to prove that the first term $D_{11}^2 L(g_0, \gamma_0)[\dot{g}, \dot{g}]$ is larger than the second term.

In the next sections, we evaluate the terms $D_{11}^2 L$ and $D_{22}^2 L$ via different methods.

3. SECOND VARIATION OF ARCLENGTH OF γ_0 IN A FAMILY OF METRICS

We briefly recall the computational scheme of [Wol89]. Let $\Phi \in \text{QD}(g_0)$ denote a quadratic differential, holomorphic with respect to a conformal metric g_0 . Then we may consider a family of metrics on S decomposed by type as

$$(3.1) \quad g_t = t\Phi dz^2 + g_0 \left(\mathcal{H}(t) + \frac{t^2|\Phi|^2}{g^2\mathcal{H}(t)} \right) dzd\bar{z} + t\bar{\Phi}d\bar{z}^2.$$

Here z is a conformal coordinate for (S, g_0) . It is straightforward to check [SY78] that the metric g_t is hyperbolic if

$$(3.2) \quad \Delta_{g_0} \log \mathcal{H}(t) = 2\mathcal{H}(t) - \frac{2t^2|\Phi|^2}{g_0^2\mathcal{H}(t)} - 2.$$

(Of course, the pullback of a hyperbolic metric by a diffeomorphism is hyperbolic, and so we might imagine that if we pullback g_t by a family of diffeomorphisms ψ_t , then the result $\psi_t^*g_t$ would also be hyperbolic. Here we have chosen a gauge by requiring that the identity map $\text{id} : (S, g_0) \rightarrow (S, g_t)$ is harmonic.)

We are interested in second variations. Differentiating twice and applying the maximum principle to the first derivative (see [Wol89] for an expanded description) yields

$$(3.3) \quad \begin{aligned} \dot{\mathcal{H}} &= \frac{d}{dt} \Big|_{t=0} \mathcal{H}(t) \equiv 0 \\ \ddot{\mathcal{H}} &= \frac{d^2}{dt^2} \Big|_{t=0} \mathcal{H}(t) = -2(\Delta - 2)^{-1} \frac{2|\Phi|^2}{g_0^2}. \end{aligned}$$

We observe that $-2(\Delta - 2)^{-1}$ is a positive operator and so $\ddot{\mathcal{H}} \geq 0$. Combining (3.1) and (3.3), we conclude that

(3.4)

$$\begin{aligned} g_t = g(t) &= g_0 dz d\bar{z} + t(\Phi dz^2 + \bar{\Phi} d\bar{z}^2) \\ &\quad + t^2/2 \left(\frac{2|\Phi|^2}{g_0^2} + -2(\Delta - 2)^{-1} \frac{2|\Phi|^2}{g_0^2} \right) g_0 dz d\bar{z} + O(t^4). \end{aligned}$$

Now use that

$$D_{11}^2 L(g_0, \gamma_0)[\dot{g}, \dot{g}] = \frac{d^2}{dt^2} \Big|_0 L(g_t, \gamma_0) = \frac{d^2}{dt^2} \Big|_{t=0} \int_{\gamma_0} \sqrt{g_t}.$$

Substituting (3.4) into this last integral with a choice of coordinate so that γ_0 is a line $\{\text{Re } z = \text{const}\}$ and differentiating under the integral symbol then yields

$$(3.5) \quad \begin{aligned} D_{11}^2 L(g_0, \gamma_0)[\dot{g}, \dot{g}] &= \int_{\gamma_0} -\frac{1}{4}(g_0)^{-3/2} (2 \text{Re } \Phi)^2 \\ &\quad + \frac{1}{2} \sqrt{g_0} \left(\frac{2|\Phi|^2}{g_0^2} - 2(\Delta - 2)^{-1} \frac{2|\Phi|^2}{g_0^2} \right) \\ &= \int_{\gamma_0} \left\{ \frac{(\text{Im } \Phi)^2}{g_0^2} - \left[2(\Delta - 2)^{-1} \frac{|\Phi|^2}{g_0^2} \right] \right\} \sqrt{g_0}, \end{aligned}$$

since $|\Phi|^2 - (\text{Re } \Phi)^2 = (\text{Im } \Phi)^2$. Both terms are positive, and so we see that this expression is positive, as we expected (and needed if the expression $D_{11}^2 L - D_{22}^2 L$ is to be positive).

Remark. As an easy model of this method, we quickly reproduce a formula for the first variation of length. (We'll have use of this expression in later sections.)

We compute the first derivative of length ℓ along $\Gamma(t)$ to be

$$(3.6) \quad \frac{d}{dt}\ell_\gamma(\Gamma(t)) = D_1L(g_t, \gamma_t)[\dot{g}] + D_2L(g_t, \gamma_t)[\dot{\gamma}].$$

Of course, as γ_0 is a geodesic, the second term $D_2L(g_t, \gamma_t)[\dot{\gamma}] = 0$ and from (3.1) and (3.2), we find

$$(3.7) \quad \begin{aligned} \frac{d}{dt}\ell_\gamma(\Gamma(t)) &= \frac{d}{dt} \int_{\gamma_0} \sqrt{g_t} \\ &= \int_{\gamma_0} \frac{\frac{d}{dt}g_t}{2g_0} \\ &= \int_{\gamma_0} \frac{\operatorname{Re} \Phi}{g_0} ds, \end{aligned}$$

by (3.4), concluding the computation.

4. SECOND VARIATION OF g_0 -ARCLENGTH OF THE FAMILY γ_t OF g_t -GEODESICS

Our next step is to evaluate the term $D_{22}^2L(g_0, \gamma_0) = \frac{d^2}{dt^2}L(g_0, \gamma_t)[\dot{\gamma}, \dot{\gamma}]$ in (2.4).

It is of course standard (see for example [Spi79b]) that if V is the variational field of a family of curves through a geodesic γ_0 , then

$$(4.1) \quad \frac{d^2}{dt^2}L(g_0, \gamma_t) = \int_{\gamma_0} \left| \frac{\partial V}{\partial s} \right|^2 - K|V|^2 ds$$

where $K = K(s)$ denotes the Gaussian curvature of the surface at the point $\gamma_0(s)$. To compare this formula (4.1) to (3.5), we will need to find an expression for V in terms of the quadratic differential Φ . That is the main goal of this section.

Of course, the defining equation of γ_t is that it is a geodesic, or equivalently that its geodesic curvature vanishes. We write this schematically, in a similar way that we write the length $L = L(g_t, \gamma_t)$, as a function $\kappa = \kappa(g_t, \gamma_t)$ of a metric and a curve:

$$(4.2) \quad \kappa(g_t, \gamma_t) = 0.$$

Differentiating in t , we find that

$$(4.3) \quad \frac{d}{dt}\kappa(g_0, \gamma_0)[\dot{\gamma}] = -\frac{d}{dt}\kappa(g_0, \gamma_0)[\dot{g}].$$

As expected, the left-hand side of (4.3) is the classical Jacobi operator $\left(\frac{d^2}{ds^2} + K\right)$, but the right hand side will involve the first derivatives of g_t , i.e. the metric g_0 and the quadratic differential Φ . Our next task will be to find an expression for the solution $\dot{\gamma}$ to (4.3).

4.1. The inhomogeneous Jacobi equation. We first expand the right hand side of (4.3). We pick conformal (Fermi) coordinates $z = x + iy$ so that the geodesic γ_0 is described by $\{x = \text{const}\}$. These coordinates are a bit unusual in that the geodesic may repeatedly visit the same points on the surface: it's possibly better to regard the geodesic as embedded in the unit tangent bundle T^1M with $z = x + iy$ its projection to the surface S .

Then, invoking a coordinate expression for κ (see [Opr04], consistent with definitions in [Spi79a] and [Spi79b]), we have

$$\frac{d}{dt}\kappa(g_0, \gamma_0)[\dot{g}] = -\frac{d}{dt} \left\{ \Gamma_{22}^1(t) \frac{\sqrt{\det g(t)}}{g_{22}(t)^{3/2}} \right\}$$

where $g(t) = g_{ij}(t)$ is defined by (2.1) as

$$(4.4) \quad g(t) = \begin{pmatrix} g_{11}(t) & g_{12}(t) \\ g_{22}(t) & g_{22}(t) \end{pmatrix} \\ = \begin{pmatrix} g_0 + 2t \operatorname{Re} \Phi & -2t \operatorname{Im} \Phi \\ -2t \operatorname{Im} \Phi & g_0 - 2t \operatorname{Re} \Phi \end{pmatrix} + O(t^2) = \begin{pmatrix} E & F \\ F & G \end{pmatrix}.$$

We include the very classical notation for the first fundamental form at the end as it actually simplifies some of our notation; for example, we write

$$(4.5) \quad \kappa(t) = -\Gamma_{22}^1(t) \frac{\sqrt{EG - F^2}}{G^{3/2}},$$

where here of course the variables $E = E(t)$, $F = F(t)$, and $G = G(t)$ all depend on t . Now, in this language, suppressing some of the dependence on t , we have

$$(4.6) \quad \Gamma_{22}^1(t) = \frac{2GF_y - GG_x - FG_y}{2(EG - F^2)}.$$

Since $\kappa(0) = 0$ and $F(0) \equiv 0$, we find that

$$(4.7) \quad \frac{\partial}{\partial x} g_{22}(0) = \frac{\partial}{\partial x} G = 0 \quad \text{on } \gamma_0.$$

We differentiate (4.6) in t and use (4.7) and $F(0) \equiv 0$ to find that

$$\begin{aligned} \frac{d}{dt} \Gamma_{22}^1(t) &= \frac{1}{2g_0^2} \{-4g_0(\operatorname{Im} \Phi)_y + 2g_0(\operatorname{Re} \Phi)_x + 2(\operatorname{Im} \Phi)(g_0)_y\} \\ &= \frac{1}{2g_0^2} \{-2g_0(\operatorname{Im} \Phi)_y + 2(\operatorname{Im} \Phi)(g_0)_y\}; \end{aligned}$$

here the last equality follows from the Cauchy-Riemann equations for the real and imaginary parts of the holomorphic quadratic differential Φ . We conclude that

$$(4.8) \quad \frac{d}{dt} \Gamma_{22}^1(t) = -\frac{\partial}{\partial y} \left\{ \frac{\operatorname{Im} \Phi}{g_0} \right\}.$$

Combining (4.3), (4.5) and (4.8) yields the equation we will focus on:

$$(4.9) \quad \frac{\partial^2}{\partial y^2} V - V = -\frac{\partial}{\partial y} \left\{ \frac{\operatorname{Im} \Phi}{g_0} \right\}.$$

Remark. It is easy to compute that the Beltrami differential tangent to our deformation is given by $\mu = \frac{\bar{\Phi}}{g_0}$. In that language, our equation (4.9) becomes

$$(4.10) \quad V_{yy} - V = \frac{\partial}{\partial y} \operatorname{Im} \mu.$$

4.2. The Primitive of the Variation Field. The right-hand side of (4.3) is the derivative of the basic quantity $-\frac{\operatorname{Im} \Phi}{g_0}$ appearing in (3.5). This term provides a link between the two different terms $D_{11}^2 L$ and $D_{22}^2 L$ of the basic expression (2.4) for the Hessian of length. To find the final formula for the Hessian of the length function, we consider the primitive of $V = \dot{\gamma}$ along γ and use this to relate the expressions for $D_{11}^2 L$ and $D_{22}^2 L = \int_{\gamma_0} V'^2 + V^2$.

In particular, we begin with the equation (4.9) and then start by defining a particular primitive U of V . The procedure is in two steps, as we need to correctly choose the constant for the primitive. So first we set

$$(4.11) \quad u(y) = \int_a^y V(s) ds$$

Note that we need to check the well-definedness of u on γ , as it is a closed loop; on the other hand it is enough to check that the period $u(2\pi) - u(0) = \int_{\gamma} V$ vanishes (here using the obvious notation for a pair of endpoints for the loop).

For convenience in the sequel, set

$$(4.12) \quad \mathcal{F} = \frac{\operatorname{Im} \Phi}{g_0}.$$

so that equation (4.9) becomes

$$(4.13) \quad V_{yy} - V = -\mathcal{F}_y$$

Then, for well-definedness of u , we observe that (letting subscripts indicate differentiation in the variable)

$$(4.14) \quad \begin{aligned} u(2\pi) - u(0) &= \int_{\gamma} V \\ &= \int_{\gamma} V_{yy} + \mathcal{F}_y \\ &= \int_{\gamma} (V_y + \mathcal{F})_y dy \\ &= 0. \end{aligned}$$

Thus u (and $u + c$, for any constant c) is well-defined along γ .

Next begin again with the equation

$$(4.15) \quad V_{yy} - V = -\mathcal{F}_y,$$

and then note that

$$(4.16) \quad \begin{aligned} (u_{yy} - u + \mathcal{F})_y &= V_{yy} - V + \mathcal{F}_y \\ &= 0. \end{aligned}$$

Thus we have that $u_{yy} - u + \mathcal{F} = c_0$, where c_0 is a constant. In particular if we set

$$(4.17) \quad U = u + c_0,$$

to be another primitive of V , then

$$(4.18) \quad U_{yy} - U = -\mathcal{F}.$$

The point of all of this is that the positive part of the second variation of length integral (4.1) will turn out to be the energy of U , while the negative part will once again be the L^2 norm of \mathcal{F} along ℓ_0 (cancelled out by a term in the metric variation contribution $D_{11}^2 L$).

We compute the contribution $-D_{22}^2 L$ from the second variation of length along the surface through

$$(4.19) \quad \begin{aligned} -D_{22}^2 L &= - \int_{\gamma_0} V_y^2 + V^2 \\ &= \int_{\gamma_0} V_{yy} V - V^2 \quad \text{by parts} \\ &= \int_{\gamma_0} (V_{yy} - V) V \\ &= \int_{\gamma_0} (-\mathcal{F}_y) V \quad \text{by (4.9)} \\ &= \int_{\gamma_0} \mathcal{F} V_y \quad \text{by parts} \\ &= \int_{\gamma_0} \mathcal{F} U_{yy} \quad \text{from the definition of } U \text{ as a primitive of } V \\ &= \int_{\gamma_0} \mathcal{F} (U - \mathcal{F}) \quad \text{from (4.18)} \\ &= - \int_{\gamma_0} \mathcal{F}^2 + \int_{\gamma_0} U \mathcal{F} \\ &= - \int_{\gamma_0} \mathcal{F}^2 + \int_{\gamma_0} U \{-(U_{yy} - U)\} \quad \text{from (4.18)} \\ (4.20) \quad &= - \int_{\gamma_0} \mathcal{F}^2 + \int_{\gamma_0} U_y^2 + U^2 \quad \text{by parts.} \end{aligned}$$

Combining this last equation with (3.5) and (4.1), we find that

$$\begin{aligned}
 \frac{d^2}{dt^2}L(g_t, \gamma_t) &= D_{11}^2L(g_0, \gamma_0)[\dot{g}, \dot{g}] - D_{22}^2L(g_0, \gamma_0)[\dot{\gamma}][\dot{\gamma}] \\
 &= \int_{\gamma_0} \mathcal{F}^2 - \left[2(\Delta - 2)^{-1} \frac{|\Phi|^2}{g_0^2} \right] - \int_{\gamma_0} V_y^2 + V^2 \\
 (4.21) \qquad &= \int_{\gamma_0} \mathcal{F}^2 - \left[2(\Delta - 2)^{-1} \frac{|\Phi|^2}{g_0^2} \right]
 \end{aligned}$$

$$\begin{aligned}
 &\quad - \int_{\gamma_0} \mathcal{F}^2 + \int_{\gamma_0} U_y^2 + U^2 \quad \text{from (4.2)} \\
 (4.22) \qquad &= \int_{\gamma_0} -2(\Delta - 2)^{-1} \frac{|\Phi|^2}{g_0^2} + \int_{\gamma_0} U_y^2 + U^2.
 \end{aligned}$$

In summary, the Weil-Petersson Hessian of length can be expressed as the sum of two integrals along the curve, each of which has a positive function as an integrand. The first integrand is the restriction to the curve of a solution of a differential equation on the surface, and the second is the energy density of a solution of a differential equation along the curve.

We record this formula as a theorem, extending Theorem 1.1 from the introduction. To set the notation, let $[\gamma]$ be the free homotopy class of a closed curve (simple or not) on the surface, and $\Gamma(t)$ a Weil-Petersson geodesic arc; the class $[\gamma]$ is represented by the $\Gamma(t)$ -geodesic γ_t . The tangent vector to Teichmüller space at $\Gamma(0)$ is given by a harmonic Beltrami differential, say $\frac{\Phi}{g_0}$. Let $\frac{\Psi}{g_0}$ denote a second harmonic Beltrami differential on $\Gamma(0)$.

Let U^Φ and U^Ψ denote the respective solutions to the ordinary differential equations (see (1.1))

$$(4.23) \qquad U_{yy} - U = -\frac{\text{Im } \Phi}{g_0}$$

and

$$(4.24) \qquad U_{yy} - U = -\frac{\text{Im } \Psi}{g_0}.$$

This is enough terminology so that we may summarize our discussion as

Theorem 4.1. *Along the Weil-Petersson geodesic arc $\Gamma(t)$, the second variation $\frac{d^2}{dt^2}\ell$ of the length $\ell(t) = L(\Gamma(t), [\gamma])$ is given by*

$$(4.25) \qquad \frac{d^2}{dt^2}\ell(t) = \int_{\gamma_0} -2(\Delta - 2)^{-1} \frac{|\Phi|^2}{g_0^2} ds + \int_{\gamma_0} [U_y^\Phi]^2 + [U^\Phi]^2 ds.$$

More generally, the Weil-Petersson Hessian $\text{Hess}L[\frac{\Phi}{g_0}, \frac{\Psi}{g_0}]$ is given by

$$(4.26) \quad \text{Hess}L\left[\frac{\Phi}{g_0}, \frac{\Psi}{g_0}\right] = \int_{\gamma_0} -2(\Delta - 2)^{-1} \frac{\text{Re } \Phi \bar{\Psi}}{g_0^2} ds + \int_{\gamma_0} U_y^\Phi U_y^\Psi + U^\Phi U^\Psi ds.$$

Proof. The solutions U^Φ and U^Ψ to (4.23) and (4.24) are unique, and the equations are linear in the unknown and the parameters Φ and Ψ . Thus the unique solution $U^{\Phi+\Psi}$ to

$$(4.27) \quad U_{yy} - U = -\frac{\operatorname{Im}(\Phi + \Psi)}{g_0}$$

satisfies

$$(4.28) \quad U^{\Phi+\Psi} = U^\Phi + U^\Psi.$$

Then a straightforward polarization of (1.2), together with our understanding (4.28), yields (4.26). \square

Remark. In terms of our previous notation for the Beltrami differential $\mu = \frac{\Phi}{g_0}$, the equation (4.23) takes the form

$$(4.29) \quad U_{yy} - U = \operatorname{Im} \mu.$$

4.3. A geometric kernel representation. The second term of equation (1.2) is expressed as the energy of the solution of a differential equation. We wish to provide a more geometric interpretation; not only do we hope that this version is more appealing on its own, but it will be important in section 9 where we treat the Thurston metric via the Hessian of the length function.

To begin, note that the second term of (1.2) (= (4.25)) may be written

$$\begin{aligned} \int_{\gamma_0} U_y^2 + U^2 ds &= - \int_{\gamma_0} U \{ (U_{yy} - U) \} ds \\ &= \int_{\gamma_0} U(s) \mathcal{F}(s) ds \end{aligned}$$

in the notation where $\mathcal{F}(s) = \operatorname{Im} \frac{\Phi}{g_0} = -\operatorname{Im} \mu$, since

$$U_{yy} - U = -\mathcal{F}$$

by (4.18). It is well-known that we can represent the solution $U(s)$ to (4.18) by

$$U(s) = - \int \mathcal{F}(t) K(s, t) dt$$

for kernels $K(s, t)$ which satisfy

$$(4.30) \quad \frac{d^2}{dt^2} K(s, t) - K(s, t) = \delta_s(t).$$

It is easy to guess the solution to (4.30) using that the solution to $\frac{d^2}{dt^2} K(s, t) - K(s, t) = 0$ are linear combinations of $\sinh(t)$ and $\cosh(t)$. (Indeed, if we represent γ_0 as the interval $[-L/2, L/2]$ with endpoints identified, set $s_0 = \pm L/2$ to be an endpoint, and look to solve (4.30) on that interval, then it is evident that setting $K(s, t) = -\cosh(t)$ is correct up to an easily computed

multiplicative constant.) In general, for γ parametrized by an interval of length L (so that we may choose $|t - s| < L/2$, we have that

$$K(s, t) = \begin{cases} -\frac{1}{2} \frac{\cosh(s-t-L/2)}{\sinh(L/2)}, & t < s \\ -\frac{1}{2} \frac{\cosh(t-s-L/2)}{\sinh(L/2)}, & t > s \end{cases}$$

solves (4.30) (where we require that $|t - s| < L/2$).

Of course, the variables s and t parametrize the curve γ_0 with respect to arclength, and so, for $|s - t| < L/2$, we have $|s - t| = d(\gamma_0(s), \gamma_0(t))$. Thus $K(s, t)$ admits the description in terms of $p = \gamma_0(s), q = \gamma_0(t)$ as

$$(4.31) \quad K(p, q) = -\frac{1}{2} \frac{\cosh(d(p, q) - L/2)}{\sinh(L/2)}.$$

This leads to the representation

$$\begin{aligned} \frac{d^2}{dt^2} \ell(t) &= \int_{\gamma_0} -2(\Delta - 2)^{-1} \frac{|\Phi|^2}{g_0^2} + \int_{\gamma_0} U_y^2(s) + U^2(s) ds \\ &= \int_{\gamma_0} -2(\Delta - 2)^{-1} \frac{|\Phi|^2}{g_0^2} - \int_{\gamma_0} U(s) \operatorname{Im} \mu(s) ds \\ &= \int_{\gamma_0} -2(\Delta - 2)^{-1} \frac{|\Phi|^2}{g_0^2} - \int_{\gamma_0} \operatorname{Im} \mu(s) \int_{\gamma_0} K(s, t) \operatorname{Im} \mu(t) dt ds \\ &= \int_{\gamma_0} -2(\Delta - 2)^{-1} \frac{|\Phi|^2}{g_0^2} - \iint_{\gamma_0 \times \gamma_0} \operatorname{Im} \mu(s) K(s, t) \operatorname{Im} \mu(t) dt ds \\ &= \int_{\gamma_0} -2(\Delta - 2)^{-1} \frac{|\Phi|^2}{g_0^2} - \iint_{\gamma_0 \times \gamma_0} \operatorname{Im} \mu(p) K(p, q) \operatorname{Im} \mu(q) ds(p) ds(q) \end{aligned}$$

(4.32)

$$\begin{aligned} &= \int_{\gamma_0} -2(\Delta - 2)^{-1} \frac{|\Phi|^2}{g_0^2} \\ &+ \frac{1}{2 \sinh(L/2)} \iint_{\gamma_0 \times \gamma_0} \operatorname{Im} \mu(p) [\cosh(d(p, q) - L/2)] \operatorname{Im} \mu(q) ds(p) ds(q). \end{aligned}$$

where $ds(p)$ and $ds(q)$ refer to arclength measure.

Part 2. Extensions and Applications of the formula for the Hessian.

5. A LOWER BOUND EXPRESSED IN TERMS OF POINTWISE QUANTITIES.

We claim

Lemma 5.1. *Let $v = 1/3 \frac{|\Phi|^2}{g_0^2}$. Then v is a subsolution of $(\Delta - 2)f = -\frac{2|\Phi|^2}{g_0^2}$ and in particular $0 \leq v \leq -2\Delta - 2)^{-1} \left(\frac{|\Phi|^2}{g_0^2} \right)$.*

We begin by noting that the curvature of a metric expressed as $G|dz|^2$ is given by

$$K(G|dz|^2) = -\frac{1}{2} \frac{1}{G} \Delta_0 \log G$$

where $\Delta_0 = \partial_x^2 + \partial_y^2$.

Then using that $K(g_0|dz|^2) \equiv -1$ and that $|\Phi_0||dz|^2$ is a flat metric with concentrated (Dirac function type) curvature singularities at the zeroes $\Phi^{-1}(0)$ of Φ , we see that

$$\begin{aligned} \Delta_0 \log \frac{|\Phi|^2}{g_0^2} &= \Delta_0 \log |\Phi|^2 - \Delta_0 \log g_0^2 \\ &= -4|\Phi|K(|\Phi||dz|^2) + 4g_0K(g_0) \\ &= 4|\Phi| \sum_{p \in \Phi^{-1}(0)} \pi \delta_p \deg_p \Phi - 4g_0, \end{aligned}$$

where δ_p indicates a delta function at p . On the other hand, using that $\Delta_0 \log F = \frac{\Delta_0 F}{F} - \frac{|\nabla_0 F|^2}{F^2}$, we see we may write

$$\Delta_0 \log \frac{|\Phi|^2}{g_0^2} = \frac{\Delta_0 \frac{|\Phi|^2}{g_0^2}}{\frac{|\Phi|^2}{g_0^2}} - \frac{|\nabla_0 \left(\frac{|\Phi|^2}{g_0^2} \right)|^2}{\left(\frac{|\Phi|^2}{g_0^2} \right)^2}.$$

Putting the last two of these equations together yields

$$\frac{1}{g_0} \Delta_0 \frac{|\Phi|^2}{g_0^2} = \frac{|\Phi|^2}{g_0^2} \left(4 \frac{|\Phi|}{g_0} \sum_{p \in \Phi^{-1}(0)} \pi \delta_p \deg_p \Phi \right) - 4 \frac{|\Phi|^2}{g_0^2} + \frac{|\nabla_0 \left(\frac{|\Phi|^2}{g_0^2} \right)|^2}{\frac{|\Phi|^2}{g_0^2} g_0}$$

In particular writing $\Delta = \frac{1}{g_0} \Delta_0$ for the g_0 -Laplace Beltrami operator on S , and noting the vanishing of the first term on the right hand side, we conclude that

$$\Delta \frac{|\Phi|^2}{g_0^2} \geq -4 \frac{|\Phi|^2}{g_0^2}.$$

We are of course interested in the operator $\Delta - 2$, so we note the obvious implication that

$$(\Delta - 2) \frac{|\Phi|^2}{g_0^2} \geq -6 \frac{|\Phi|^2}{g_0^2}$$

so that $v = 1/3 \frac{|\Phi|^2}{g_0^2}$ is a subsolution for the equation $(\Delta - 2)f = -2 \frac{|\Phi|^2}{g_0^2}$.

It is obvious that $v = \frac{1}{3} \frac{|\Phi|^2}{g_0^2} \geq 0$, and if f satisfies $(\Delta - 2)f = -2 \frac{|\Phi|^2}{g_0^2}$, then $\Delta(f - v) \leq 2(f - v)$ and so the minimum principle guarantees that at a minimum of $(f - v)$, we have $f - v \geq 0$; hence $f - v \geq 0$ everywhere, concluding the proof of the lemma. \square

Combining Lemma 5.1 with Theorem 1.1 we obtain

Corollary 5.2. *Let $\Phi \in \text{QD}(g_0)$ be a holomorphic quadratic differential in (Σ, g_0) and let $\Gamma(t)$ denote a Weil-Petersson geodesic arc with initial tangent vector given by the harmonic Beltrami differential $\bar{\Phi}g_0^{-1}$. Let $\ell(t)$ denote the geodesic length of a representative γ_t of a curve class $[\gamma]$ on S . Then, for γ_0 the geodesic represented of $[\gamma]$ on $\Gamma(0)$, we have*

$$\left. \frac{d^2}{dt^2} \right|_{t=0} \ell(t) \geq \frac{1}{3} \int_{\gamma_0} \frac{|\Phi|^2}{g_0^2} ds. \quad \square$$

We will apply this estimate in the section 8.

6. THE SECOND VARIATION OF THE LENGTH OF AN ARC

In this section, we adapt our derivation to the case where S is a surface of finite genus with a finite number of punctures, and we are interested in the variation of length of an arc α that runs between two of the punctures (or a puncture itself). Naturally, the length of such an arc is infinite, so we will be discussing the variation of some regularization of its length; nevertheless, all of the basic considerations will extend to this case with only minor modifications.

6.1. Notation and preliminaries. Let α_t be the geodesic on (S, g_t) that connects punctures p and q in a fixed homotopy class (rel p and q). Consider a sequence of points $p_n, q_n \in \alpha_0$ with $p_n \rightarrow p$ and $q_n \rightarrow q$ and let $\alpha_{t,n}$ denote the finite length g_t -geodesic arc connecting p_n to q_n which is homotopic (rel p_n, q_n) to $\alpha_{0,n} \subset \alpha_0$. Our plan is to derive a formula for

$$\frac{d^2}{dt^2} L(g_t, \alpha_{t,n}),$$

show that the limit exists and is independent of the choice of the sequence $\{p_n, q_n\}$.

We learned while preparing this manuscript that Wolpert [Wol07] recently treated the analogous case of finite length arcs between horocycles.

It is easy to check that the formal preliminaries remain the same as in the derivation of (2.4), and so we conclude

$$(6.1) \quad \frac{d^2}{dt^2} L(g_t, \alpha_{t,n}) = D_{11}^2 L(g_0, \alpha_{0,n})[\dot{g}, \dot{g}] - D_{22}^2 L(g_0, \alpha_{0,n})[\dot{\alpha}_{0,n}, \dot{\alpha}_{0,n}].$$

6.2. The Second Variation of Arclength of $\alpha_{0,n}$ in g_t . As in the case of a closed curve, the first term in (6.1) is relatively straightforward to compute; the only new issue to consider is the dependence of the term on the choice of endpoints p_n, q_n of $\alpha_{t,n}$. Indeed, exactly as in the derivation of (3.5), we formally compute

$$(6.2) \quad D_{11}^2 L(g_0, \alpha_{0,n}) = \int_{\alpha_{0,n}} \left\{ \frac{|\text{Im } \Phi|^2}{g_0^2} - 2(\Delta - 2)^{-1} \frac{|\Phi|^2}{g_0^2} \right\} \sqrt{g_0}$$

where the principal issue is to determine the meaning of $(\Delta - 2)^{-1} \frac{|\Phi|^2}{g_0^2}$.

6.3. Variations of metrics of finite area. The basic point here in understanding $-2(\Delta - 2)^{-1} \frac{|\Phi|^2}{g_0^2}$ is to construe it as $\ddot{\mathcal{H}}$ for the family of pullback metrics g_t in (3.1). As these maps $\text{id} : (S, g_0) \rightarrow (S, g_t)$ are harmonic, we can apply some results from the theory of harmonic maps between cusped hyperbolic surfaces.

In this direction, results in [Wol91] (Theorem 5.1) and of Lohkamp (see the remark after Theorem 4 in [Loh91], especially with Lemma 12 informed by Proposition 3.13 in [Wol91]) proved that $\mathcal{H}(t) \in C^{k,\alpha}(S, g_0)$ was analytic in t on the compactified surface \bar{M} . In particular $\ddot{\mathcal{H}}$ is bounded.

Indeed, we can easily show from this that $\ddot{\mathcal{H}} = O(\frac{1}{(\log \frac{1}{r})^\alpha}) = O(y^{-\alpha})$ for some $\alpha \in (0, 1)$ as $r \rightarrow 0$. The basic elements of this argument is that $\frac{1}{(\log \frac{1}{r})^\alpha} = y^{-\alpha}$ for some $\alpha \in (0, 1)$ is a supersolution of the equation $(\Delta - 2)\ddot{\mathcal{H}} = -2\|\Phi\|^2$ on the cusp, as well as the point that the kernel of $(\Delta - 2)$ on a half-infinite cylinder $\mathcal{C} = \{\text{Im } z > 1, |\text{Re } z| < 1/2\}$ (with the standard identifications) is spanned by the pair of functions $k_1(z) = y^2$ and $k_2(z) = y^{-1}$. With that background, consider on the finite cylinder $\{1 < \text{Im } z < y_n\}$, a function $H_j(z)$ of the form $H_j(z) = C_0 y^{-\alpha} + C_1 y^{-1} + \epsilon_j y^2$.

Then for appropriate choices of $\epsilon_j \rightarrow 0$, we find that $H_j(z)$ majorizes $\ddot{\mathcal{H}}$; letting $j \rightarrow \infty$ and $\epsilon_j \rightarrow 0$ while C_0 and C_1 stay bounded (as the boundary values $\ddot{\mathcal{H}}(z)$ for $\{\text{Im } z = 1\}$ are fixed independently of j) allows us to conclude that $\ddot{\mathcal{H}}(z)$ decays like $C_0 y^{-\alpha} + C_1 y^{-1}$. Thus $\ddot{\mathcal{H}}(z) = O(y^{-\alpha}) = O(\frac{1}{(\log \frac{1}{r})^\alpha})$.

Looking ahead to the final form of the second variation of length, it is worth recording the

Proposition 6.1. *Let $p_n \rightarrow p$ and $q_n \rightarrow q$, and let $\alpha_{0,n}$ be the geodesic arc connecting p_n to q_n as in the introduction to the section. Then*

$$\int_{\alpha_{0,n}} -2(\Delta - 2)^{-1} \frac{|\Phi|^2}{g_0^2} ds = \int_{\alpha_{0,n}} \frac{1}{2} \ddot{\mathcal{H}} ds$$

converges as $n \rightarrow \infty$.

Proof. In the upper half plane coordinates, we have that for each end of an $\alpha_{0,n}$,

$$(6.3) \quad \int_{\alpha_n} \ddot{\mathcal{H}} ds = \int_a^{y_n} \ddot{\mathcal{H}} \frac{dy}{y} = \int_a^{y_n} O\left(\frac{1}{y^\alpha}\right) \frac{dy}{y} = O(1) \quad \text{as } n \rightarrow \infty.$$

The proposition then follows from the integrals being positive. \square

6.4. The Jacobi Field for an Arc. The next term we must address is the second term in (6.1). The variational vector field, defined geometrically, also satisfies (4.9). Here, of course the variational field V_n depends on n , as it is defined in terms of $\alpha_{t,n}$; our notation is meant to reflect that. The form of the second variation of arclength is also unchanged at $\int_{\alpha_{0,n}} V_{n,y}^2 + V_n^2$, as the boundary term vanishes once we require the family of curves $\alpha_{t,n}$ to have p_n and q_n as endpoints, independently of t .

The primitive $U_n = \int_a^y V_n(s)ds$ exists as before — in fact, in the setting of an open arc, there is not a well-definedness issue to check, though we still have to adjust by a constant. In particular, note that for $\hat{U}_n = \int_{p_n}^y V_n(s)ds$ (so that $\hat{U}_n = \int_{p_n}^y V_n''(s) - (\operatorname{Im} \mu)'(s)ds$), we have $\hat{U}_n'' - \hat{U}_n = \operatorname{Im} \mu + (V_n'(p_n) - \operatorname{Im} \mu(p_n))$. So set $U_n = \int_{p_n}^y V_n(s)ds + (V_n'(p_n) - \operatorname{Im} \mu(p_n))$. Then U_n solves the (boundary value) problem

$$(6.4) \quad \begin{aligned} U_n'' - U_n &= \operatorname{Im} \mu \\ U_n(p_n) &= V_n'(p_n) - \operatorname{Im} \mu(p_n) \\ U_n(q_n) &= V_n'(q_n) - \operatorname{Im} \mu(q_n). \end{aligned}$$

The last boundary condition follows after applying the fundamental theorem of calculus to

$$\begin{aligned} U_n(q_n) &= \int_{p_n}^{q_n} V_n(s)ds + V_n'(p_n) - \operatorname{Im} \mu(p_n) \\ &= \int_{p_n}^{q_n} V_n''(s) - (\operatorname{Im} \mu)'(s)ds + V_n'(p_n) - \operatorname{Im} \mu(p_n) \end{aligned}$$

We will soon show that these boundary values, though non-zero, tend to zero and have no effect on the limiting relation.

At this stage, it is useful to write a kernel representation for V_n . Let $L_n = d(p_n, q_n)$, and parametrize the arc $\alpha_{0,n}$ by $[-\frac{L_n}{2}, \frac{L_n}{2}]$. It will not ultimately affect the results if in these coordinates the midpoints of $[p_n, q_n]$ (represented in our coordinates by the origin) do not remain in a compact set.

The kernel for the operator $\frac{d^2}{dy^2} - 1$ on the segment $[-\frac{L_n}{2}, \frac{L_n}{2}]$ is given by

$$(6.5) \quad K_n(y, s) = \begin{cases} -\frac{\sinh(\frac{L_n}{2}+s) \sinh(\frac{L_n}{2}-y)}{\sinh(L_n)}, & -\frac{L_n}{2} \leq s \leq y \leq \frac{L_n}{2} \\ -\frac{\sinh(\frac{L_n}{2}-s) \sinh(\frac{L_n}{2}+y)}{\sinh(L_n)}, & -\frac{L_n}{2} \leq y \leq s \leq \frac{L_n}{2}. \end{cases}$$

This gives the representations

$$(6.6) \quad V_n(y) = \int_{p_n}^{q_n} K_n(y, s)(\operatorname{Im} \mu)'(s)ds$$

and

$$(6.7) \quad U_n(y) = \int_{p_n}^{q_n} K_n(y, s)(\operatorname{Im} \mu)(s)ds + a_n \cosh(y) + b_n \sinh(y),$$

where a_n and b_n are chosen to satisfy the boundary conditions in (6.4).

We will estimate the asymptotics of these boundary terms in the next subsection, but we display the preliminaries for that analysis here, in the present context of integral formulas for the relevant geometric objects.

One of the terms in the boundary condition is given by V'_n at the corresponding boundary point. For example, $V'_n(p_n)$ is given by

$$(6.8) \quad V'_n(p_n) = \int_{p_n}^{q_n} \frac{d}{dy} \Big|_{y=p_n^+} K_n(y, s) (\operatorname{Im} \mu)'(s) ds.$$

We then compute that

$$(6.9) \quad \frac{d}{dy} K_n(y, s) = \begin{cases} \frac{\sinh(\frac{L_n}{2} + s) \cosh(\frac{L_n}{2} - y)}{\sinh(L_n)}, & -\frac{L_n}{2} \leq s \leq y \leq \frac{L_n}{2} \\ -\frac{\sinh(\frac{L_n}{2} - s) \cosh(\frac{L_n}{2} + y)}{\sinh(L_n)}, & -\frac{L_n}{2} \leq y \leq s \leq \frac{L_n}{2} \end{cases}.$$

Combining (6.8) and (6.9) yields the representation

$$(6.10) \quad V'_n(p_n) = - \int_{-\frac{L_n}{2}}^{\frac{L_n}{2}} \frac{\sinh(\frac{L_n}{2} - s)}{\sinh(L_n)} (\operatorname{Im} \mu)'(\alpha_n(s)) ds.$$

The final matter is the analogue of the derivation of (4.2) from the variation of arlength. This involves three integrations by parts, and so we need to consider the boundary terms from each integration; however, all of the terms have either V_n or $U_{n,y} = V_n$ as a factor, and so all vanish at the endpoints p_n and q_n .

We conclude that

$$(6.11) \quad -D_{22}^2 L(g_0, \alpha_{0,n})(\dot{\alpha}_{0,n}, \dot{\alpha}_{0,n}) = - \int_{\alpha_{0,n}} \mathcal{F}^2 + \int_{\alpha_{0,n}} U_{n,s}^2 + U_n^2$$

in the notation of (4.12).

We combine formulae (6.2) and (6.11) for the derivatives $D_{11} L(g_0, \alpha_{0,n})$ and $D_{22} L(g_0, \alpha_{0,n})$ to find

$$(6.12) \quad \frac{d^2}{dt^2} L(g_t, \alpha_{t,n}) = \int_{\alpha_{0,n}} -2(\Delta - 2)^{-1} \frac{|\Phi|^2}{g_0^2} + \int_{\alpha_{0,n}} U_{n,s}^2 + U_n^2 ds.$$

6.5. Passage to the limit. Naturally, we are interested in taking the limit of (6.12) as $n \rightarrow \infty$. That the first term converges in the content of Proposition 6.1.

For the second term, we need to estimate the asymptotics of $U_{n,y} = V_n$ and U_n . We claim

Proposition 6.2. *The fields V_n converge to a field V defined on the entire arc α_0 . The primitives U_n may be chosen so that, not only does U_n converge to a field U with $U_s = V$, but also*

$$(6.13) \quad \int_{\alpha_{0,n}} U_{n,s}^2 + U_n^2 \longrightarrow \int_{\alpha_0} U_s^2 + U^2 < \infty.$$

Here U_s and $U_{n,s}$ indicate derivatives of U and U_n with respect to the arlength parameter s .

Proof. It is useful to begin with an observation.

Lemma 6.3. *Near an end of (S, g_0) with coordinates from $\{|z| < 1\}$, we have $|\mu| = O(r(\log \frac{1}{r})^2)$ and $|\frac{d}{ds}\mu| = O(r(\log \frac{1}{r})^3)$.*

Proof. In the coordinate disk, the holomorphic quadratic differential $\Phi = \frac{c}{z} + h.o.t$, while $\frac{1}{g_0} = r^2(\log \frac{1}{r})^2$ and $\frac{d}{ds} = r \log \frac{1}{r} \frac{\partial}{\partial r}$ along radial geodesics. The estimates are then immediate. \square

Recall the terms a_n and b_n in (6.7) that adjust the solution U_n for the inhomogeneous boundary conditions. Our next goal is to prove that these adjustments are asymptotically inconsequential.

Lemma 6.4. $a_n, b_n = o(e^{-c\frac{L_n}{2}})$.

Proof. The goal is to show that the boundary conditions given in (6.4) are small. In that case, since we would have $a_n \cosh(y) + b_n \sinh(y) = o(1)$ for $y = \pm \frac{L}{2}$ from (6.7), we easily see the statement of the lemma.

There are two types of terms in the expression (6.4): those given by $|(\operatorname{Im} \mu)(p_n)|$ and $|(\operatorname{Im} \mu)(q_n)|$, and those given by $V'_n(p_n)$ and $V'_n(q_n)$. We will treat them separately.

We first claim that $|(\operatorname{Im} \mu)(p_n)|$ and $|(\operatorname{Im} \mu)(q_n)|$ decay as $n \rightarrow \infty$. This of course easily follows from Lemma 6.3.

It is only slightly more difficult to use (6.10) to show that $V'_n(p_n)$ and $V'_n(q_n)$ decay to zero as $n \rightarrow \infty$. To see this, note that for any fixed y^* , we have $\int_{y^*}^{\frac{L_n}{2}} \frac{\sinh(\frac{L_n}{2} - s)}{\sinh(L_n)} ds = O(e^{-\frac{L_n}{2} - y^*})$, and that $|(\operatorname{Im} \mu)'$ is bounded on all of the geodesic arc α_0 , while vanishing into the cusp by Lemma 6.3. (Note also that $\int_{-\frac{L_n}{2}}^{\frac{L_n}{2}} \frac{\sinh(\frac{L_n}{2} - s)}{\sinh(L_n)} ds \leq 1$.) These preliminaries are enough to estimate $V'_n(p_n)$ (with an analogous argument for $V'_n(q_n)$).

We begin from (6.10) with

$$\begin{aligned} |V'_n(p_n)| &\leq \int_{-\frac{L_n}{2}}^{y^*} \frac{\sinh(\frac{L_n}{2} - s)}{\sinh(L_n)} |(\operatorname{Im} \mu)'(\alpha_n(s))| ds \\ &\quad + \int_{y^*}^{\frac{L_n}{2}} \frac{\sinh(\frac{L_n}{2} - s)}{\sinh(L_n)} |(\operatorname{Im} \mu)'(\alpha_n(s))| ds \\ &\leq \max_{[p_n, y^*]} |(\operatorname{Im} \mu)'| \int_{-\frac{L_n}{2}}^{y^*} \frac{\sinh(\frac{L_n}{2} - s)}{\sinh(L_n)} ds \\ &\quad + \max_{\alpha_0} |(\operatorname{Im} \mu)'| \int_{y^*}^{\frac{L_n}{2}} \frac{\sinh(\frac{L_n}{2} - s)}{\sinh(L_n)} ds. \end{aligned}$$

Then, for ϵ small, use Lemma 6.3 to pick y^* so that $\max_{[p_n, y^*]} |(\operatorname{Im} \mu)'| \leq \epsilon$. Then the first term is bounded by ϵ while the second term is bounded by $\max_{\alpha_0} |(\operatorname{Im} \mu)'| O(e^{-\frac{L_n}{2} - y^*})$. Letting $L_n \rightarrow \infty$ and $-\frac{L_n}{2} \leq y^* \rightarrow -\infty$ somewhat more slowly than $-\frac{L_n}{2} \rightarrow -\infty$ (e.g. $y^* = -\frac{L_n}{4}$) shows that $V'_n(p_n) \rightarrow 0$. \square

With these preliminaries, we easily conclude the proof of the proposition. As $L_n \rightarrow \infty$, we find that for fixed y , the kernels $K_n(y, s)$ limit on

$$K(y, s) = \begin{cases} -\frac{e^{s-y}}{2}, & s \leq y \\ -\frac{e^{y-s}}{2}, & y \leq s \end{cases}$$

which we can write succinctly as

$$K(y, s) = -\frac{1}{2}e^{-d(y,s)}$$

As both $(\operatorname{Im} \mu)(s)$ and $(\operatorname{Im} \mu)'(s)$ decay while $K_n(y, s)$ converge to an integrable function, we see that the formulas (6.6) and (6.7) show uniform convergence of V_n to a well-defined variation field V , as well as primitives U_n to a well-defined (primitive) function U . As $U'_n = V_n$, the uniform convergence of $\{U_n\}$ and $\{V_n\}$ show that $U' = V$. Using Lemma 6.4, we obtain the formula

$$\begin{aligned} U(y) &= \int_p^q K(y, s)(\operatorname{Im} \mu)(s) ds. \\ (6.14) \quad &= \int_p^q \frac{1}{2}e^{-d(y,s)}(\operatorname{Im} \mu)(s) ds \end{aligned}$$

Next, it is easy to see that $U(y) \rightarrow 0$ as either $y \rightarrow p$ or $y \rightarrow q$. Mimicking the argument that showed that $V'_n(p_n) \rightarrow 0$, choose y^* close enough to p so that $\max_{[p, y^*]} |\mu| < \frac{\epsilon}{2}$. Then choose y even deeper into the cusp so that $d(y, y^*) \geq -\log(\frac{\epsilon}{\max_{\alpha_0} |\mu|})$. Then

$$\begin{aligned} |U(y)| &\leq \left| \int_p^{y^*} K(y, s)(\operatorname{Im} \mu)(s) ds \right| + \left| \int_{y^*}^q K(y, s)(\operatorname{Im} \mu)(s) ds \right| \\ &< \frac{\epsilon}{2} \int_p^{y^*} |K(y, s)| ds + \max_{\alpha_0} |\mu| \int_{y^*}^{\infty} \frac{1}{2} e^{-d(y,s)} ds \\ &\leq \frac{\epsilon}{2} + \frac{1}{2} e^{-d(y, y^*)} \max_{\alpha_0} |\mu| \\ &\leq \epsilon \end{aligned}$$

as desired.

Finally, we address the finiteness of the energy of U . Of course, the metric g_0 near the cusp point p (or q) may be expressed as $g_0 = |z|^{-2}(\log |z|)^{-2}|dz|^2$, and so the estimates for μ may be written as $|\mu(y)| = O(e^{y-e^y})$. It is then an easy estimate that

$$\int_{\alpha_0} U'^2 + U^2 = \frac{1}{2} \iint_{\alpha_0 \times \alpha_0} e^{-|s-y|} \operatorname{Im} \mu(s) \operatorname{Im} \mu(y) ds dy < \infty$$

and is the limit of $\int_{\alpha_{0,n}} U'^2 + U^2$.

This concludes the proof of the Proposition. \square

We summarize our discussion in this section with formulae for the second variations of an open arc α analogous to those in Theorem 1.1 for the second variations of length of a simple closed curve γ .

Theorem 6.5. *Along the Weil-Petersson geodesic arc $\Gamma(t)$, the second variation $\frac{d^2}{dt^2}\ell$ of the $\Gamma(t)$ -length $\ell(t) = L(\Gamma(t), \alpha)$ of a (class of an) arc α is given by the (convergent) expression*

(6.15)

$$\begin{aligned} \frac{d^2}{dt^2}\ell(t) &= \int_{\alpha} -(\Delta - 2)^{-1} \frac{2|\Phi|^2}{g_0^2} ds + \int_{\alpha} [U_y^\Phi]^2 + [U^\Phi]^2 ds \\ (6.16) \quad &= \int_{\alpha} -(\Delta - 2)^{-1} \frac{2|\Phi|^2}{g_0^2} ds + \frac{1}{2} \iint_{\alpha_0 \times \alpha_0} e^{-d(s,y)} \operatorname{Im} \mu(s) \operatorname{Im} \mu(y) ds dy. \end{aligned}$$

More generally, the Weil-Petersson Hessian $\operatorname{Hess}L[\frac{\bar{\Phi}}{g_0}, \frac{\bar{\Psi}}{g_0}]$ is given by the (convergent) expression

$$(6.17) \quad \operatorname{Hess}L[\frac{\bar{\Phi}}{g_0}, \frac{\bar{\Psi}}{g_0}] = \int_{\alpha} -(\Delta - 2)^{-1} \frac{2 \operatorname{Re} \Phi \bar{\Psi}}{g_0^2} ds + \int_{\alpha} U_y^\Phi U_y^\Psi + U^\Phi U^\Psi ds$$

$$(6.18) \quad \begin{aligned} &= \int_{\alpha} -(\Delta - 2)^{-1} \frac{2 \operatorname{Re} \Phi \bar{\Psi}}{g_0^2} ds \\ &\quad + \frac{1}{2} \iint_{\alpha_0 \times \alpha_0} e^{-d(s,y)} \operatorname{Im} \mu(s) \operatorname{Im} \nu(y) ds dy, \end{aligned}$$

where $\mu = \frac{\bar{\Phi}}{g_0}$ and $\nu = \frac{\bar{\Psi}}{g_0}$ are the harmonic Beltrami differential representatives of two tangent directions at $[g_0]$.

Here U^Φ satisfies the equation (1.1) along the arc γ , a condition which forces $U^\Phi \rightarrow 0$ along γ as it tends to the punctures p and q .

7. CONVEXITY FOR LAMINATIONS

We have already seen in Theorem 1.1 and Corollary 5.2 that if γ is a simple closed curve, then on a Weil-Petersson ray $\Gamma = \Gamma(t)$, we have

$$(7.1) \quad \frac{d^2}{dt^2}\ell_\gamma(\Gamma(t)) \geq \frac{1}{3} \int_{\gamma} \|\Phi_t\|^2 ds$$

where $\Phi(t)$ is the holomorphic quadratic differential tangent to Γ at $\Gamma(t)$, and $\|\Phi\| = \frac{|\Phi|}{g_0}$. In this section, we extend that result to prove the

Proposition 7.1. *Let $\Gamma = \Gamma(t)$ be a Weil-Petersson ray and λ a measured lamination on S . Then*

$$(7.2) \quad \frac{d^2}{dt^2}\ell_\lambda(\Gamma(t)) \geq \frac{1}{3} \int_{\lambda} \|\Phi_t\|^2 ds$$

where Φ_t is the holomorphic quadratic differential tangent to Γ at $\Gamma(t)$.

Our definition of $\int_\lambda \|\Phi\|^2 ds$ is straightforward and parallels the definition of length of the lamination λ . See [Bon01] for a background discussion. In particular, a measured lamination λ is defined as a measure $\lambda(k) = \int_k d\lambda$ on transverse arcs k . Choose arcs k_1, \dots, k_J which are transverse to λ and construct flow boxes $\{F_i\}$ for λ bounded by the k_j and parallel to λ .

Then if G_λ is the geodesic lamination underlying λ , then $\lambda - \cup_j k_j$ is a (possibly infinite) collection of finite length arcs. The length of λ is then the integral, with respect to the transverse measure $d\lambda$ of λ , of the lengths of the finite arc components. More precisely, we lift each of these components λ to $\hat{\lambda}_a \subset T^1M$, endow $\hat{\lambda}_a$ with the natural arclength measure ds , and then integrate the product to get

$$\ell(\lambda) =: \int_\lambda ds =: \iint_{\lambda_a} ds d\lambda(a).$$

In order to define $\int_\lambda \|\Phi\|^2 ds$, we proceed analogously, except that we note that the function $\|\Phi\|^2$ on S then naturally defines a measure $\|\Phi\|^2 ds$ on T^1M . In other words, we set

$$\int_\lambda \|\Phi\|^2 ds = \iint_{\lambda_a} \|\Phi\|^2 ds d\lambda(a).$$

Proof of Proposition 7.1. Let γ_n be a sequence of simple closed curves converging to λ . The idea is to apply (7.1) to γ_n and then take a limit in n to find (7.2).

Now ℓ_γ and ℓ_λ are real analytic functions on Γ [Ker85], and thus since $\ell_{\gamma_n} \rightarrow \ell_\lambda$, so does $\frac{d^2}{dt^2} \ell_{\gamma_n} \rightarrow \frac{d^2}{dt^2} \ell_\lambda$. Thus the left-hand sides of (7.1) converge to the left-hand side of (7.2).

For the right-hand side, the argument is virtually tautological. We first note that the arclength measure $\|\Phi\|^2 ds$ is continuous on T^1M . Then, choose n sufficiently large so that the flow boxes $\{F_i\}$ described above also serve as flow boxes for γ_n . Then the definition of convergence in \mathcal{ML} then easily implies that the right-hand side of (7.1) converges to the right-hand side of (7.2). \square

8. CONVEXITY OF $\ell^{\frac{1}{2}}$ AND UPPER BOUNDS ON THE HESSIAN

8.1. The function $\ell^{\frac{1}{2}}$.

8.1.1. *The function $\ell^{\frac{2}{3}}$.* In light of (7.1), we can quickly refine the basic convexity result for the length ℓ to a convexity result for a concave function of ℓ , namely $\ell^{\frac{2}{3}}$. We will see in the next subsection that this is not sharp, but at this stage it is elementary.

To begin, recall from (3.7) that the first variation of length may be expressed as

$$\frac{d}{dt} \ell_\gamma(\Gamma(t)) = \int_{\gamma_0} \frac{\operatorname{Re} \Phi}{g_0} ds$$

We can then estimate this derivative as

$$\begin{aligned} \left| \frac{d}{dt} \ell_\gamma(\Gamma(t)) \right| &\leq \int_{\gamma_0} \left| \frac{\operatorname{Re} \Phi}{g_0} \right| ds \\ &\leq \left(\int_{\gamma_0} \frac{|\Phi|^2}{g_0^2} ds \right)^{\frac{1}{2}} \ell_{\gamma_0}^{\frac{1}{2}} \end{aligned}$$

Squaring and combining with (7.1) yields

$$(8.1) \quad \ell \frac{d^2}{dt^2} \ell_\gamma(\Gamma(t)) \geq \frac{1}{3} \left(\frac{d}{dt} \ell_\gamma(\Gamma(t)) \right)^2.$$

We compute $\frac{d^2}{dt^2} \ell^{\frac{2}{3}}(\Gamma_t)$ and substitute in the above inequality to conclude the

Corollary 8.1. *The function $\ell^{\frac{2}{3}}$ is Weil-Petersson convex on Teichmüller space.*

8.1.2. *The length of an annulus.* In this subsection, we compute in two ways the second variation of the length of the core geodesic in a hyperbolic annulus; the basic result is well-known (see Example 3.6 in [Wol08]).

let $\mathcal{C}(\ell)$ denote a complete hyperbolic annulus whose core geodesic has length ℓ . This cylinder may be parametrized as

$$\mathcal{C}(\ell) = \left[-\frac{\pi}{2\ell}, \frac{\pi}{2\ell} \right] \times [0, 1],$$

where top and bottom edges are identified, and we consider the metric cylinder as equipped with the hyperbolic metric $g = ds_\ell^2 = \ell^2 \csc^2 \ell x |dz|^2$.

Consider rotationally the harmonic map $R(t) : \mathcal{C}(\ell) \rightarrow \mathcal{C}(\ell + t)$ which does not twist the boundary; in other words, this map may be expressed in coordinates as $R(t) = u(t) + iv(t)$ where $u(t)(z) = u(t)(x)$ and $v(t)(z) = y$. Now the rotationally invariant holomorphic quadratic differentials on a cylinder have a particularly simple form: we may write one as $\Phi = cdz^2$ in the complex coordinates above. Thus, taking one of these as a Hopf differential for our map, and using that the holomorphic energy \mathcal{H} , the Beltrami differential ν and the Hopf differential Φ may be related by $\Phi = g(\ell_0)\mathcal{H}\bar{\nu}$, we conclude that

$$u' = \frac{1 + \nu}{1 - \nu} = \frac{\ell^2 + c \cos^2 \ell x}{\ell^2 - c \cos^2 \ell x}.$$

If $c(t)$ is the factor so that the Hopf differential is parametrizing a family of harmonic maps $R(t) : \mathcal{C}(\ell) \rightarrow \mathcal{C}(\ell + t)$ whose targets are progressing through Teichmüller space $\mathcal{T}(\mathcal{C})$ at unit Weil-Petersson speed, then two conditions hold: (i) the choice of $c(t)$ provides for $c(t)dz^2$ to be the Hopf differential for the map $R(t)$, and (ii) $\left\| \frac{d}{dt} \Phi(t) \right\|_{WP} = 1$.

Now, for $w(t)$ to have image $\mathcal{C}(\ell + t)$, we must have the boundary of $\mathcal{C}(\ell)$ map to the boundary of $\mathcal{C}(\ell + t)$, i.e.

$$\frac{\pi}{2(\ell + t)} = u\left(\frac{\pi}{2\ell}\right) = \int_0^{\frac{\pi}{2\ell}} u'(x) dx + u(0) = \int_0^{\frac{\pi}{2\ell}} \frac{1 + \nu}{1 - \nu} = \frac{\ell^2 + c \cos^2 \ell x}{\ell^2 - c \cos^2 \ell x} dx.$$

Upon differentiating in t and finding the resulting elementary integrals, we obtain $\dot{c} = -\ell$. Thus

$$\left\| \frac{\partial}{\partial \ell} \right\|_{WP}^2 = \|\dot{c}g_\ell^{-1}\|_{WP} = \|(-\ell)\ell^{-2} \cos^2 \ell x\|_{WP} = \frac{\pi^2}{\ell}$$

after another explicit integration. Thus $\left\| \frac{\partial}{\partial \ell} \right\|_{WP} = \frac{\pi}{\ell^{\frac{1}{2}}}$ and so $ds_{WP}^2 = \frac{\pi^2}{\ell} d\ell^2$ on the Teichmüller space $\mathcal{T}(\mathcal{C})$. This implies that on this space, $\frac{d\ell}{ds} = \pi^{-1} \ell^{\frac{1}{2}}$ and so $\ell = (2\pi)^{-2} s^2$. Thus the length ℓ of the core geodesic satisfies that $\ell^{\frac{1}{2}}$ is convex, but not convex to any lower power.

Remark. Alternatively, we may use the formulas (1.2) and (3.7) to analytically find the same result. In that case, if we set $\Phi = cdz^2$, then $\ddot{\ell} = \int_{\gamma_0} -2(\Delta - 2)^{-1} \{c^2 g^{-4}\}$. However, the equation $\Delta u - 2u = c^2 g^{-4}$ reduces in this case to an ordinary differential equation, whose solution we can require to be bounded on the boundary. It is then elementary (see the analogous analysis in (8.18)ff using the method of variations of parameters) to find an exact expression for u on γ_0 . We then compare with the expression in (3.7) for the first derivative to obtain the half-power convexity result above: this confirms, at least in this very simple case, the formula (1.2).

8.1.3. $\ell^{\frac{1}{2}}$ is convex. We now offer a geometric argument of Wolpert's recent result [Wol08] that $\ell_\gamma^{\frac{1}{2}}$ is Weil-Petersson convex.

Comparison of lifted harmonic map to rotationally invariant map. The essential point is best understood in the setting of the annular covers $(\mathcal{C}, \tilde{g}_t)$ of the family of surfaces (S, g_t) . Consider the harmonic maps $w_t : (S, g_0) \rightarrow (S, g_t)$ and their lifts $\tilde{w}_t : (S, \tilde{g}_0) \rightarrow (S, \tilde{g}_t)$. These lifts are in the homotopy class of the rotationally invariant harmonic map $R_t : \mathcal{C}(\ell_0) \rightarrow \mathcal{C}(\ell_t)$, where $\ell_\gamma(g_t) = \ell_t$. Now w_t is conformal only at the (isolated) zeroes of the Hopf differential, and so, off of small neighborhoods of the zeroes of the lifted Hopf differential, the harmonic map \tilde{w}_t has quasi-isometric constant uniformly bounded away from 1. By contrast, one can either compute or reason geometrically that the rotationally invariant harmonic map $R_t : \mathcal{C}(\ell_0) \rightarrow \mathcal{C}(\ell_t)$ has quasi-isometric constant tending uniformly to 1 as one leaves compacta in $\mathcal{C}(\ell_0)$: the image curves are growing exponentially in length, so energy efficiency requires the map to be increasingly close to an isometry as one leaves compact sets.

Let $\mathcal{H}^R(t)$ be the holomorphic energy (see equations (3.1)-(3.3)) of $R(t)$ and $\mathcal{H}(t)$ be the holomorphic energy of \tilde{w}_t . Since both $R(t)$ and \tilde{w}_t are the identity when $t = 0$, and since, as we have just seen, $R(t)$ is asymptotically an isometry while \tilde{w}_t is boundedly away from being the identity off small sets, then we find

$$\ddot{\mathcal{H}}(t) \geq \ddot{\mathcal{H}}^R(t)$$

outside some large compact set (at least away from small neighborhoods of the zeroes of the lift $\tilde{\Phi}$ of Φ). In particular, parametrizing $\mathcal{C}(\ell)$ as in

subsection 8.1.2, we see that

$$(8.2) \quad \int_{x=\pm\frac{\pi}{2\ell}\mp\delta} \ddot{\mathcal{H}}(t) \geq \int_{x=\pm\frac{\pi}{2\ell}\mp\delta} \ddot{\mathcal{H}}^R(t).$$

A comparison of ODEs. The rest of the proof follows by applying other inequalities that reflect that $R(t)$ is a harmonic map of lower (regularized in some way) energy than \tilde{w}_t . In particular, consider the Fourier expansion $\Phi = \sum b_n(x)e^{2\pi iny}$ of the quadratic differential Φ (where the map \tilde{w}_t has Hopf differential Φ).

Because

$$\dot{\ell} = \int_{x=0} \frac{\operatorname{Re} \Phi}{g} \sqrt{g} dy = \ell \operatorname{Re} b_0$$

we know that the Hopf differential Φ^R for the rotationally invariant map $R(t)$ must be $\Phi^R = \operatorname{Re} b_0 dz^2$ on $\mathcal{C}(\ell)$: this is because the targets $\mathcal{C}(\ell + t)$ agree for the two maps w_t and $R(t)$, and hence the change in core-curve length is the same.

The upshot is that, for an arbitrary constant curvature circle $\{x = \xi\}$, we have

$$(8.3) \quad \int_{x=\xi} \frac{|\Phi|^2}{g^2} ds = g^{-\frac{3}{2}}(x) \sum |b_n|^2 \geq g^{-\frac{3}{2}}(x) (\operatorname{Re} b_0)^2 = \int_{x=\xi} \frac{|\Phi^R|^2}{g^2} ds.$$

Of course, we know from formula (1.2) that

$$\ddot{\ell} \geq \int_{\gamma_0} -2(\Delta - 2)^{-1} \frac{|\Phi|^2}{g^2} ds = \frac{1}{2} \int_{\gamma_0} \ddot{\mathcal{H}} ds$$

To estimate this last integral, let u be the solution of

$$\Delta_g u - 2u = \frac{-2|\Phi|^2}{g^2}.$$

If we were to integrate this equation along the vertical parameter curves $\{x = \text{const}\}$, we would obtain an ordinary differential equation for the function $\int_x u dy = \int_x \frac{1}{2} \ddot{\mathcal{H}} dy$ in the single variable $x \in (-\frac{\pi}{2\ell}, \frac{\pi}{2\ell})$, i.e.

$$\frac{1}{g} \partial_x^2 \int_x u dy - 2 \int_x u dy = -2 \int_x \frac{|\Phi|^2}{g^2} dy$$

Of course, a similar equation holds for the integrals $\int_x \frac{1}{2} \ddot{\mathcal{H}}^R dy$ and $\int_x \frac{|\Phi^R|^2}{g^2} dy$ associated to the rotationally invariant map. Indeed, inequality (8.2) says that there are boundary points $x = \pm\frac{\pi}{2\ell} \mp \delta$ at which $\int_x \frac{1}{2} \ddot{\mathcal{H}}^R dy \leq \int_x \frac{1}{2} \ddot{\mathcal{H}} dy$; moreover, inequality (8.3) asserts that on the interval $(-\frac{\pi}{2\ell}, \frac{\pi}{2\ell})$, the right-hand-side of (8.1.3) is less in the lifted case than in the rotationally invariant case.

The upshot is that the comparison principle for ordinary differential equations implies that

$$\begin{aligned} \int_x -2(\Delta - 2)^{-1} \frac{|\Phi|^2}{g^2} dy &= \int_x \frac{1}{2} \ddot{\mathcal{H}}^R dy \\ &\geq \int_x \frac{1}{2} \ddot{\mathcal{H}} dy \\ &= \int_x -2(\Delta - 2)^{-1} \frac{(\operatorname{Re} b_0)^2}{g^2} dy. \end{aligned}$$

In particular, specializing to the curve $\{x = 0\}$, and recalling the implication above of (1.2), we find

$$\ddot{\ell} \geq \int_{\gamma_0} -2(\Delta - 2)^{-1} \frac{|\Phi|^2}{g^2} \geq \int_{\gamma_0} -2(\Delta - 2)^{-1} \frac{(\operatorname{Re} b_0)^2}{g^2} = \ddot{\ell}^R \geq 2 \frac{\dot{\ell}^2}{\ell}.$$

Here the last inequality is inherited from the rotationally invariant case: recall that our choice that $\operatorname{Re} \int_{\gamma_0} \Phi = \int_{\gamma_0} \Phi^R$ implies that the infinitesimal change of lengths agree between the lifted and rotationally invariant maps. We conclude

Corollary 8.2. (*Wolpert [Wol08]*) *The function $\ell^{\frac{1}{2}}$ is Weil-Petersson convex in the Teichmüller space $\mathcal{T}(S)$.*

8.2. A general upper bound for the Hessian. We have already seen a lower bound for the Hessian of length in Corollary 5.2 and Proposition 7.1. In this passage, we note an easy upper bound as well.

We begin by noting that if

$$(\Delta - 2)h = -2 \frac{|\Phi|^2}{g_0^2}$$

on a surface S , then the maximum principle implies

$$(8.4) \quad h \leq \left\| \frac{|\Phi|^2}{g_0^2} \right\|_{\infty}$$

where the right hand side is the maximum of the function $\frac{|\Phi|^2}{g_0^2}$ on S . In the formula (1.2), this will estimate the first term.

To estimate the second term, we consider equation (4.18) (combined with (4.12))

$$(8.5) \quad U_{yy} - U = -\frac{\operatorname{Im} \Phi}{g_0}.$$

The maximum principle then implies that

$$(8.6) \quad U \leq \max_{\gamma} \left| \frac{\operatorname{Im} \Phi}{g_0} \right|.$$

Thus the second term in formula (1.2) is estimated as

$$\begin{aligned}
 (8.7) \quad \int_{\gamma} U_y^2 + U^2 &= - \int_{\gamma} (U_{yy} - U)U \\
 &= \int_{\gamma} \left(\frac{\operatorname{Im} \Phi}{g_0} \right) U \\
 &\leq \int_{\gamma} \left(\max_{\gamma} \left| \frac{\operatorname{Im} \Phi}{g_0} \right| \right) \left(\max_{\gamma} \left| \frac{\operatorname{Im} \Phi}{g_0} \right| \right) \quad \text{after a substitution} \\
 &= \ell_{\gamma} \left(\max_{\gamma} \left| \frac{\operatorname{Im} \Phi}{g_0} \right| \right)^2.
 \end{aligned}$$

We conclude, taking into account Corollary 5.2, that

Corollary 8.3.

$$(8.8) \quad \frac{1}{3} \int_{\gamma} \|\Phi_t\|^2 ds \leq \frac{d^2}{dt^2} \ell_{\gamma}(\Gamma(t)) \leq \ell_{\gamma} \left(\max_S \left\| \frac{|\Phi|^2}{g_0^2} \right\| + \left(\max_{\gamma} \left| \frac{\operatorname{Im} \Phi}{g_0} \right| \right)^2 \right).$$

8.3. Estimates for the Weil-Petersson connection near the compactification divisor. In this passage we refine the method above to estimate the Weil-Petersson connection on a codimension two distribution $\mathcal{P} \subset T\mathcal{M}$ of the tangent bundle near the Deligne-Mumford compactification divisor (i.e. for surfaces with small injectivity radius) which is in some sense “parallel” to the tangent bundle of the compactification divisor. Roughly, we prove that \mathcal{P} is quite flat, in the sense that for $X, Y \in \mathcal{P}$, we will have that the normal component $(\nabla_X Y)^{\perp}$ of $(\nabla_X Y)$ satisfies $(\nabla_X Y)^{\perp} = O(\ell^2)$. (Here ℓ signifies the length of the curve which vanishes on the nearby component of the compactification divisor.

To state this precisely, we choose a simple closed curve $\gamma \subset S$ and a small number $\ell > 0$; we consider the level set $L_{\gamma}(\ell)$ of hyperbolic surfaces for which $L(\cdot, \gamma) = \ell$. The set $L_{\gamma}(\ell)$ is a submanifold of the Teichmüller space \mathcal{T} of real codimension one, and it is orthogonal to the vector $\operatorname{grad} \ell_{\gamma}$, the Weil-Petersson gradient of ℓ_{γ} . Let J be the almost complex structure of \mathcal{T} and consider the projection $\tau \in TL_{\gamma}(\ell)$ of $J \operatorname{grad} \ell_{\gamma}$ into $TL_{\gamma}(\ell)$. Let $\mathcal{P} \subset TL_{\gamma}(\ell)$ denote the distribution of $(\dim \mathcal{T} - 2)$ -planes in $TL_{\gamma}(\ell)$ orthogonal to the span of $\operatorname{grad} \ell_{\gamma}$ and τ ; note that \mathcal{P} is Weil-Petersson orthogonal to both $\operatorname{grad} \ell_{\gamma}$ and $J \operatorname{grad} \ell_{\gamma}$.

Remark. One does not expect \mathcal{P} to be integrable. In particular, one does not expect that $J \operatorname{grad} \ell_{\gamma}$ is parallel to the Fenchel-Nielsen twist vector field. (See [Wol82].) Nevertheless, it will be a consequence of equation (8.10) of the early part of the next proof that, as $\ell_{\gamma} \rightarrow 0$, the distribution \mathcal{P} converges to the tangent bundle TC_{γ} of the compactification divisor $\mathcal{C}_{\gamma} = \{\ell_{\gamma} = 0\}$ of the augmented Teichmüller space $\bar{\mathcal{T}}$. (Compare [Wol91].)

Now, for $X \in \mathcal{P}$ and Y a section of $\mathcal{P} \rightarrow \mathcal{T}$, (i.e. a vector field on $L_{\gamma}(\ell)$), we can consider the Weil-Petersson covariant derivative $\nabla_X Y$. Of course, the vector $\nabla_X Y$ has components both in \mathcal{P} and in the orthogonal

complement \mathcal{P}^\perp of \mathcal{P} ; we focus here on the component of this vector in the orthogonal complement.

Remark. It might be difficult to formulate general results on the full vector $\nabla_X Y$ in useful and incisive way. For example, if we were to “lift” a curve α from the compactification divisor \mathcal{C}_γ to an almost parallel curve $\hat{\alpha}$ tangent to \mathcal{P} , then since $\nabla_{\hat{\alpha}} \dot{\alpha} \subset T\mathcal{C}_\gamma$ can be arbitrary, so might we expect $\nabla_{\hat{\alpha}} \dot{\hat{\alpha}}$ to have an arbitrary non-orthogonal component.

Our main result in this section is

Theorem 8.4. *In the notation above, for $X \in \mathcal{P}$ and $Y \in \mathcal{P}$ of unit norm, we have*

$$(8.9) \quad (\nabla_X Y)^\perp = \langle \nabla_X Y, \mathcal{P}^\perp \rangle = O(\ell_\gamma^2).$$

In particular, $\langle \nabla_X Y, \text{grad } \ell_\gamma \rangle = O(\ell_\gamma^2)$ and $\langle \nabla_X Y, J \text{grad } \ell_\gamma \rangle = O(\ell_\gamma^2)$.

Remark. Similar results were obtained recently by Wolpert [Wol08], using his estimates on the Hessian. From the formula for the Hessian presented in (1.2), we see in the present derivation the elementary nature of the expansion of $\nabla_X Y$ in ℓ_γ . The flatness of order $O(\ell_\gamma^2)$ occurs because the (explicit) rotationally invariant even solution $u(x) = x \tan \ell x + 1/\ell$ of the Jacobi equation has the following property: at the core geodesic of the cylinder, this function $u(x)$ is smaller by a factor comparable to $O(\ell_\gamma)$ than it is on the boundary of the cylinder. Since the geodesic has length $O(\ell_\gamma)$, the dominant term of the integral of $u(x)$ over the geodesic decays like $O(\ell_\gamma^2)$.

Proof. We begin the proof of Theorem 8.4 with a preliminary proposition characterizing the quadratic differentials which represent elements of \mathcal{P} .

Proposition 8.5. *Let $X \in T_{[M,g]} \mathcal{T}$ with $X \in \mathcal{P}$ and $\Phi = \Phi_X$ be a holomorphic quadratic differential on the surface (S, g) for which $\bar{\Phi}/g$ is a harmonic Beltrami differential representing X . Then for γ the geodesic on S used to define $L_\gamma(\ell)$, we have*

$$(8.10) \quad \int_\gamma \frac{\Phi}{g} ds = 0.$$

Proof. Since $X \in \mathcal{P}$, we have $\langle X, \text{grad } \ell_\gamma \rangle = 0$. But from (3.7), we find that

$$\begin{aligned} 0 &= \langle X, \text{grad } \ell_\gamma \rangle \\ &= X(\ell_\gamma) \\ &= \int_\gamma \frac{\text{Re } \Phi}{g} ds. \end{aligned}$$

Of course the other defining condition of $X \in \mathcal{P}$ is that $\langle X, J \text{grad } \ell_\gamma \rangle = 0$. But since the Weil-Petersson metric is Kähler, this implies

$$\begin{aligned} 0 &= \langle X, J \text{grad } \ell_\gamma \rangle \\ &= - \langle JX, \text{grad } \ell_\gamma \rangle. \end{aligned}$$

But JX is represented by $i\bar{\Phi}/g$, and so we derive as above that

$$0 = \int_{\gamma} \frac{\operatorname{Re} i\Phi}{g} ds,$$

proving the result. □

Remark. The covectors in $T^*\mathcal{C}_{\gamma}$ are represented by holomorphic quadratic differentials which have simple poles at the nodes. On the other hand, a generic covector in $T_{\Sigma}^*\bar{\mathcal{I}}$ at an element $\Sigma \in \mathcal{C}_{\gamma}$ has a second order pole with a non-vanishing residue at the node obtained by pinching γ .

Continuation of the Proof of Theorem 8.4 The heart of the matter is a computation of $\operatorname{Hess}l_{\gamma}(X, X)$, in particular to prove

Lemma 8.6. *For $X \subset \mathcal{P}$ as above, we have*

$$(8.11) \quad \operatorname{Hess}l_{\gamma}(X, X) = O(\ell_{\gamma}^2).$$

To see that this is enough, note first that polarization will imply then that $\operatorname{Hess}l_{\gamma}(X, Y) = O(\ell_{\gamma}^2)$ for $X, Y \subset \mathcal{P}$ of unit norm. But then

$$\begin{aligned} - \langle \nabla_X Y, \operatorname{grad} l_{\gamma} \rangle &= -(\nabla_X Y)l_{\gamma} \\ &= (XY - \nabla_X Y)l_{\gamma} \quad \text{since } X, Y \text{ are tangent to } L_{\gamma}(\ell) \\ &= \operatorname{Hess}l_{\gamma}(X, Y) \quad \text{by definition} \\ (8.12) \quad &= O(\ell_{\gamma}^2). \quad \text{by Lemma 8.6} \end{aligned}$$

Moreover

$$\begin{aligned} \langle \nabla_X Y, J \operatorname{grad} l_{\gamma} \rangle &= - \langle J \nabla_X Y, \operatorname{grad} l_{\gamma} \rangle \\ &\quad \text{as } J \text{ is a Weil-Petersson isometry} \\ &= - \langle \nabla_X JY, \operatorname{grad} l_{\gamma} \rangle \\ &\quad \text{as Weil-Petersson is a Kähler metric} \\ &= - \langle \nabla_X Z, \operatorname{grad} l_{\gamma} \rangle \end{aligned}$$

for some $Z \in \mathcal{P}$ as \mathcal{P} is J -invariant, being orthogonal to a J -invariant subspace of a Kähler manifold. Then $\langle \nabla_X Y, J \operatorname{grad} l_{\gamma} \rangle = O(\ell_{\gamma}^2)$ follows in the manner of (8.12). This concludes the proof of Theorem 8.4, pending the proof of the main lemma. □

Proof of Lemma 8.6. The basic idea of the proof is to estimate the terms in the formula (1.2) for the Hessian, where we take X to be represented by a harmonic Beltrami differential $\mu = \bar{\Phi}/g$, and we apply the features of $X \subset \mathcal{P} \subset TL_{\gamma}(\ell)$ to prove that those terms are small. In particular, because the length $L(g, \gamma)$ is small, the geodesic γ is embedded in a wide, thin collar. Then, by Proposition 8.5, we learn that $|\Phi|$ must decay rapidly towards the center of the collar. Those facts together are enough to conclude that each of the pair of terms in (1.2) is small.

We carry out the plan in steps.

Step 0. The collar. Consider the collar $\mathcal{C} = [-\frac{1}{\ell} \sec^{-1} \frac{1}{\ell}, \frac{1}{\ell} \sec^{-1} \frac{1}{\ell}] \times [0, 1]$ with horizontal edges $[-\frac{1}{\ell} \sec^{-1} \frac{1}{\ell}, \frac{1}{\ell} \sec^{-1} \frac{1}{\ell}] \times \{0, 1\}$ identified, equipped with the hyperbolic metric $g_0 = \ell^2 \sec^2 \ell x |dz^2|$. This collar \mathcal{C} embeds in a neighborhood of the geodesic γ , with $\{0\} \times [0, 1]$ mapping onto γ .

Step 1. Decay of $|\Phi|$. On this annular collar \mathcal{C} , we may regard the quadratic differential Φ as a function (or more formally, we divide Φ by the nonvanishing holomorphic quadratic differential dz^2 to obtain a function in the quotient). Then, by the rotational invariance of the collar (or by working in Fermi coordinates), we see that g may be taken as constant along γ , and so the conditions in Proposition 8.5 imply that Φ has no period in the collar. Finally, we estimate boundary conditions.

Since, for ℓ small, every $X \subset \mathcal{P} \subset TL_\gamma(\ell)$ can be approximated on compacta away from γ by an integrable meromorphic quadratic differential on a noded Riemann surface in the compactification divisor \mathcal{C}_γ , and the collection of such quadratic differentials of unit Weil-Petersson norm is compact, we see that we may take $|\Phi|$ as bounded on the horocycle of length one on $\partial\mathcal{C}$.

Because Φ is holomorphic and hence harmonic, the vanishing of the period together with the fixed boundary conditions is enough to show that $|\Phi|$ decays rapidly on the interior of the cylinder.

We can obtain an estimate of this decay through a Fourier analysis of Φ . On the cylinder \mathcal{C} , set $\Phi = \sum a_n(x) e^{2\pi i n y}$. Then the harmonicity of Φ implies

$$0 = \Delta\Phi = \sum (a_n''(x) - 4\pi n^2 a_n(x)) e^{2\pi i n y}$$

and in particular the equations

$$a_n''(x) - 4\pi n^2 a_n(x) = 0.$$

Now as $a_0(0) = \int_{x=0} \Phi = 0$ by (8.10), we find that $\int_{\gamma^*} \Phi = 0$ along any cycle γ^* in \mathcal{C}_γ homologous to γ , hence $a_0(x) = 0$.

Of course, $\int_{\partial\mathcal{C}_\gamma} |\Phi|^2 \leq C_0$ by our argument on limits above, and so

$$(8.13) \quad \sum_{n \neq 0} a_n^2(\pm \frac{1}{\ell} \sec^{-1} \frac{1}{\ell}) = \int_{\partial\mathcal{C}_\gamma} |\Phi|^2 \leq C_0.$$

We note that

$$(8.14) \quad \begin{aligned} \left(\sum_{n \neq 0} a_n^2(x) \right)'' &= \sum_{n \neq 0} 2a_n'' a_n + 2(a_n')^2 \\ &= \sum_{n \neq 0} 8\pi^2 n^2 a_n^2 + 2(a_n')^2 \quad \text{by (8.3)} \\ &\geq 8\pi^2 \sum_{n \neq 0} a_n^2. \end{aligned}$$

Thus by the maximum principle applied to the differential inequality (8.14) with boundary conditions (8.13), we have

$$(8.15) \quad \int_{x=x_0} |\Phi|^2 \leq \frac{C_0 \cosh \sqrt{8\pi} x_0}{\cosh \sqrt{8\pi} (\frac{1}{\ell} \sec^{-1} \frac{1}{\ell})} := D \cosh \sqrt{8\pi} x_0.$$

Finally consider $\Phi(z_0)$, for $z_0 = x_0 + iy_0$, a fixed point of the collar. Note that on our parametrization of the collar, the balls of radius $\frac{1}{2}$ inject into the parameter domain. As Φ is harmonic,

$$\begin{aligned} |\Phi(z_0)| &\leq \frac{4}{\pi} \left| \int_{B_{\frac{1}{2}}(z_0)} \Phi \right| \\ &\leq \frac{4}{\sqrt{\pi}} \left(\int_{B_{\frac{1}{2}}(z_0)} |\Phi|^2 \right)^{\frac{1}{2}} \\ &< \frac{4}{\sqrt{\pi}} \left(\int_{|\operatorname{Re}(z-z_0)| \leq \frac{1}{2}} |\Phi|^2 \right)^{\frac{1}{2}} \\ &= \frac{4}{\sqrt{\pi}} \left(\int_{\operatorname{Re} z_0 - \frac{1}{2}}^{\operatorname{Re} z_0 + \frac{1}{2}} \int_{x=t} |\Phi|^2 dy dt \right)^{\frac{1}{2}} \\ &\leq \frac{4}{\sqrt{\pi}} \left(\int_{\operatorname{Re} z_0 - \frac{1}{2}}^{\operatorname{Re} z_0 + \frac{1}{2}} D \cosh \sqrt{8\pi} t dt \right)^{\frac{1}{2}} \quad \text{by (8.15)} \\ &= D_1 (\cosh \sqrt{8\pi} x_0)^{\frac{1}{2}} \end{aligned}$$

We conclude that

$$|\Phi(z_0)|^2 < D_2 \cosh \sqrt{8\pi} x_0,$$

where (using (8.15))

$$(8.16) \quad D_2 = O(e^{-\frac{\sqrt{8\pi}}{\ell}})$$

in ℓ , for ℓ small, justifying our remark about the decay of $|\Phi(z)|$ into the collar.

Step 2: the function U . There are two terms in the formula (1.2) for the Hessian; here we estimate the one involving the energy of the function U . In particular, we already know from (8.7) that

$$\begin{aligned} \int_{\gamma_0} U'^2 + U^2 &\leq \ell (\max \frac{|\Phi|}{g})^2 \\ &\leq \ell g^{-2}|_{x=0} D_2 \\ &= O(\ell^{-3} e^{-\frac{\sqrt{8\pi}}{\ell}}) \end{aligned}$$

using $g|_{x=0} = \ell^2$ and (8.16). This term is then consistent with the statement of the lemma that $\operatorname{Hess} \ell(X, X) \leq O(\ell^2)$.

Step 3: the function $(\Delta - 2)^{-1} \frac{|\Phi|^2}{g^2}$. We are left to estimate the second term in the expression (1.2) for $\text{Hess}^\ell(X, X)$, namely

$$\int_\gamma -2(\Delta - 2)^{-1} \frac{|\Phi|^2}{g^2}.$$

In particular, we need to estimate the solution u_0 to the equation

$$(8.17) \quad (\Delta - 2)u_0 = -2 \frac{|\Phi|^2}{g^2}.$$

evaluated on the core geodesic.

We again estimate the solution to this by estimating it on the cylinder \mathcal{C}_γ , on which we have good control on $\frac{|\Phi|^2}{g^2}$. Of course we already know from the maximum principle (see (8.4)) that

$$u_0 \leq \sup \frac{|\Phi|^2}{g^2}.$$

Now the latter is bounded \mathcal{C}_γ , and the complement of \mathcal{C}_γ has bounded geometry on which $\int_{M \sim \mathcal{C}_\gamma} \frac{|\Phi|^2}{g^2} dA_g \leq 1$ (because Φ is of unit Weil-Petersson norm). We conclude that there is a C_0 for which

$$\frac{|\Phi|^2}{g^2} \Big|_{\partial \mathcal{C}_\gamma} \leq C_0$$

Thus by the maximum principle, it is enough to estimate the solution u of the boundary value problem

$$\begin{aligned} \frac{1}{g} u''(x) - 2u(x) &= \frac{D_2 \cosh \sqrt{8\pi} x}{g^2} \\ u\left(\pm \frac{1}{\ell} \sec^{-1} \frac{1}{\ell}\right) &= C_0 \end{aligned}$$

Using that $g = \ell^2 \sec^2 \ell x$, we rewrite the equation above as

$$(8.18) \quad \begin{aligned} u''(x) - 2\ell^2 \sec^2 \ell x u(x) &= D_3 \cosh(\sqrt{8\pi} x) \cos^2 \ell x \\ u\left(\pm \frac{1}{\ell} \sec^{-1} \frac{1}{\ell}\right) &= C_0, \end{aligned}$$

where $D_3 = O(\ell^{-2} e^{-\sqrt{8\pi}/\ell})$. The homogeneous equation

$$u''(x) - 2\ell^2 \sec^2 \ell x u(x) = 0$$

has the two solutions

$$(8.19) \quad \begin{aligned} u_1(x) &= \tan \ell x \\ u_2(x) &= x \tan \ell x + \frac{1}{\ell}. \end{aligned}$$

Using these, one can solve for a particular solution of the form $u_0 = u_1v_1 + u_2v_2$ by elementary integrations, namely

$$\begin{aligned} v_1 &= \int u_2 D_3 \cosh(\sqrt{8\pi}x) \cos^2 \ell x \\ v_2 &= - \int u_1 D_3 \cosh(\sqrt{8\pi}x) \cos^2 \ell x. \end{aligned}$$

An asymptotic expansion shows that $u_1v_1 + u_2v_2|_{\partial\mathcal{C}_\gamma} = O(1)$; this is actually quite remarkable, as both u_1v_1 and u_2v_2 are separately comparable to ℓ^{-3} on $\partial\mathcal{C}_\gamma$. Moreover, $u_1v_1 + u_2v_2|_{x=0} = O(e^{-\sqrt{8\pi}/\ell}\ell^{-2})$. With these computations in mind, we observe that the general solution to (8.18) is given by

$$u = c_1u_1 + c_2u_2 + u_1v_1 + u_2v_2.$$

By our estimates on $u_1v_1 + u_2v_2|_{\partial\mathcal{C}_\gamma}$ and the definitions of u_1 and u_2 , we see that we may take $c_1 = O(\ell)$ and $c_2 = O(\ell^2)$. Thus we compute that

$$\begin{aligned} u(0) &= c_1u_1(0) + c_2u_2(0) + u_1(0)v_1(0) + u_2(0)v_2(0) \\ &= c_2u_2(0) + u_2(0)v_2(0) \quad \text{since } u_1(0) = 0 \\ &= O(\ell) + O(e^{-\sqrt{8\pi}/\ell}\ell^{-3}) \\ &= O(\ell). \end{aligned}$$

We conclude that $-2(\Delta - 2)^{-1}\frac{|\Phi|^2}{g^2}\Big|_\gamma = u_0\Big|_\gamma \leq u(0) = O(\ell)$ and so

$$\begin{aligned} \int_\gamma -2(\Delta - 2)^{-1}\frac{|\Phi|^2}{g^2} &= \int_\gamma u ds \leq O(\ell) \int_\gamma ds \\ &= O(\ell^2). \end{aligned}$$

Combining the estimate for this term of the Hessian with the estimate for the other term of the Hessian discussed in Step 2 concludes the proof of the lemma. \square

9. THE THURSTON METRIC AND THE WEIL-PETERSSON METRIC

From the formula (1.2), we can easily derive the result [Wol86b] (see also [McM08] and [Bon88]) that the Thurston metric is a multiple of the Weil-Petersson metric.

9.1. The Thurston Metric. We begin by recalling the Thurston metric. To define this, imagine a sequence $\{\gamma_n\}$ of closed curves which are becoming equidistributed in the sense that if B is a ball in the unit tangent bundle T^1S , and we lift γ_n to its representative in T^1S , then

$$\lim_{n \rightarrow \infty} \frac{\ell(\gamma_n \cap B)}{\ell(\gamma_n)} = \frac{\text{Volume}(B)}{\text{Volume}(T^1S)}.$$

Thurston noted that since, for such a sequence of curves, $\frac{d\ell(\gamma_n)}{\ell(\gamma_n)} \rightarrow 0$ as $n \rightarrow \infty$, then $\frac{\text{Hess}\ell(\gamma_n)}{\ell(\gamma_n)}$ would tend to a symmetric quadratic form on $T(S)$; by the convexity of the length function, this tensor would be positive semi-definite, hence a (pseudo)-metric. Wolpert showed that

Theorem 9.1. [Wol86b]. *The Thurston metric is a multiple of the Weil-Petersson metric.*

The goal of the present section is to give a proof of this result that proceeds from evaluating formula (1.2) on a sequence $\{\gamma_n\}$ of curves that are becoming equidistributed in T^1S .

9.2. First Variation. We begin by first showing that $d\ell(\gamma_n)/\ell(\gamma_n) \rightarrow 0$ as $n \rightarrow \infty$. Recall from (3.7) that

$$\frac{d}{dt}L(g_t, \gamma_{n,t}) = \int_{\gamma_n(0)} \frac{\text{Re } \Phi}{g_0} ds.$$

Now, the curves γ_n have unit tangent vectors $\frac{d}{ds}\gamma_n(s)$ which equidistribute themselves in T^1S , and so

$$\begin{aligned} \frac{1}{\ell(\gamma_n)} \frac{d}{dt}L(g_t, \gamma_{n,t}) &= -\frac{1}{\ell(\gamma_n)} \int_{\gamma_n} \frac{\text{Re } \Phi}{g_0} ds \\ &= -\int_{\gamma_n} \frac{\text{Re } \Phi}{g_0} \frac{ds}{\ell(\gamma_n)} \\ &\rightarrow \iiint_{T^1S} \frac{\text{Re } \Phi(p, \theta)}{g_0} \frac{d \text{vol}_{T^1S}(p, \theta)}{\text{vol}(T^1S)} \end{aligned}$$

where we interpret the meaning of the notation as follows. In the discussion so far, we have written $\text{Re } \Phi$ to denote the value of the expression $\text{Re } \varphi$, when the quadratic differential $\Phi = \varphi dz^2$ was written in coordinates $z = x + iy$ with $\frac{\partial}{\partial y}$ being tangent to the geodesic. Now the vector field $\frac{\partial}{\partial y}$ lifts to the canonical vector fields in T^1S tangent to the geodesic flow. We let the expression $\text{Re } \Phi(p, \theta)$ denote the value of $\text{Re } \Phi$ on the surface in terms of a coordinate $z = x + iy$ in which the geodesic direction described by $(p, \theta) \in T^1S$ is in the coordinate direction $\frac{\partial}{\partial y}$. Of course, if we change surface coordinates so that $z_\theta = e^{-i\theta}z$, then for $\Phi = \varphi_\theta dz_\theta^2$, we have $\varphi_\theta = e^{2i\theta}\varphi_0$. Thus, when we integrate along the fiber of $T^1S \rightarrow S$ (with respect to θ in the coordinates (z, θ) for T^1S), we find $\int \frac{\text{Re } \varphi(p, \theta)}{g_0} d\theta = 0$.

9.3. The second variation. We recall the (second) formula (1.2) for the second variation of length:

$$\begin{aligned}
 & \frac{1}{\ell(\gamma_n)} \frac{d^2}{dt^2} \ell(\gamma_n(t)) \\
 &= \frac{1}{\ell(\gamma_n)} \int -(\Delta - 2)^{-1} \frac{2|\Phi|^2}{g_0^2} ds \\
 & \quad + \frac{1}{\ell(\gamma_n)} \iint_{\gamma_n \times \gamma_n} \operatorname{Im} \mu(p) \frac{\cosh(d(p, q) - \frac{\ell(\gamma_n)}{2})}{2 \sinh(\frac{\ell(\gamma_n)}{2})} \operatorname{Im} \mu(q) ds(p) ds(q) \\
 &= \frac{1}{\ell(\gamma_n)} \int -2(\Delta - 2)^{-1} \frac{|\Phi|^2}{g_0^2} ds \\
 & \quad + \frac{1}{\ell(\gamma_n)} \int_{\gamma_n} \frac{\operatorname{Im} \Phi(p)}{g_0(p)} \int_{\gamma_n} \frac{\cosh(d(p, q) - \ell(\gamma_n)/2)}{2 \sinh(\ell(\gamma_n)/2)} \frac{\operatorname{Im} \Phi(q)}{g_0(q)} dq dp \\
 &= I_1 + I_2.
 \end{aligned}$$

We examine the two integrals I_1 and I_2 separately. The first integral is immediate, as

$$I_1 = \int_{\gamma_n} -2(\Delta - 2)^{-1} \frac{|\Phi|^2}{g_0^2} \frac{ds}{\ell(\gamma_n)} \longrightarrow 2\pi \iint_M -2(\Delta - 2)^{-1} \frac{|\Phi|^2}{g_0^2} \frac{d \operatorname{Area}}{2\pi \operatorname{Area}}$$

by the equidistribution property, and

$$\iint_M -2(\Delta - 2)^{-1} \frac{|\Phi|^2}{g_0^2} \frac{d \operatorname{Area}}{\operatorname{Area}} = \iint \{-2(\Delta - 2)^{-1}(1)\} \frac{|\Phi|^2}{g_0^2} \frac{d \operatorname{Area}}{\operatorname{Area}}$$

as the operator $(\Delta - 2)^{-1}$ is self adjoint. Then as $-2(\Delta - 2)^{-1}(1) = 1$, we find that

$$(9.1) \quad I_1 \longrightarrow \iint \frac{|\Phi|^2}{g_0^2} \frac{d \operatorname{Area}}{\operatorname{Area}}.$$

9.4. The Integral I_2 . We turn next to I_2 , where the computation is a bit more involved. Our basic plan mirrors our discussion of the first variation; we extend the terms in the integrand of I_2 to all of T^1S , and then integrate over the circular fiber to be left with an integral over the surface.

We begin our more detailed discussion of the second (energy) integral by considering the version (4.33) of it in terms of a geometric kernel, i.e.

$$I_2 = \frac{1}{\ell(\gamma_n)} \int_{\gamma_0} \frac{\operatorname{Im} \Phi(p)}{g_0(p)} \int_{\gamma_0} \frac{\cosh(d(p, q) - L/2)}{2 \sinh(L/2)} \frac{\operatorname{Im} \Phi(q)}{g_0(q)} dq dp$$

where $L = \ell(\gamma_0)$.

Now considering γ_0 as an embedded curve in the unit tangent bundle $T^1(S, g_0)$, if we fix the point $p = (\bar{p}, v) \in T^1S$ as representing a point $\bar{p} \in S$

and a unit vector $v \in T_p^1 S$, then a point $q = (\bar{q}, w)$ along γ_0 at distance t from p could be written

$$q = \exp_{\bar{p}} tv = G_t p$$

where G_t denotes the geodesic flow in $T^1 S$ for distance t .

Thus we see that as $L \rightarrow \infty$, the integral I_2 converges to

$$\lim_{L \rightarrow \infty} I_2 = \int_{T^1 S} \frac{\operatorname{Im} \Phi(p)}{g_0(p)} \int_0^\infty \frac{e^{-t}}{2} \left(\frac{\operatorname{Im} \Phi(G_t p)}{g_0(G_t(p))} + \frac{\operatorname{Im} \Phi(G_{-t} p)}{g_0(G_{-t}(p))} \right) \frac{dt dp}{\operatorname{vol}(T^1 S)}.$$

Of course, as p varies in the fiber $\{(\bar{p}, e^{i\theta} v)\}$ over $\bar{p} \in S$, we observe the points $G_t(\bar{p}, v)$ also arising as $G_{-t}(\bar{p}, -v)$, and so we may rewrite the above limit integral as

$$\lim_{L \rightarrow \infty} I_2 = \int_{T^1 S} \frac{\operatorname{Im} \Phi(p)}{g_0(p)} \int_0^\infty e^{-t} \frac{\operatorname{Im} \Phi(G_t p)}{g_0(G_t(p))} \frac{dt dp}{\operatorname{vol}(T^1 S)}.$$

To evaluate this last integral, imagine $\bar{p} = 0$ in the disk $\{|z| < 1\}$ and we represent the hyperbolic metric as

$$g_0 = \frac{4|dz|^2}{(1-r^2)^2} = g_0(0)(1-r^2)^{-2}|dz|^2.$$

An important matter here (as it was in the calculation of the first variation) is the question of how to interpret the meaning of $\operatorname{Im} \Phi(q)$ in these coordinates: recall that we understood $\operatorname{Im} \frac{\Phi}{g_0}$ to be the value of $\operatorname{Im} \frac{\Phi}{g_0}$ when we defined the geodesic γ_0 as a vertical line in the coordinate system. In the fixed coordinate system of the disk $\{|z| < 1\}$, write $\Phi = \phi(z)dz^2$; then since the hyperbolic geodesic through the origin and the point $z = re^{i\theta}$ is given by the ray $te^{i\theta}$, we see that $\operatorname{Im} \Phi|_{re^{i\theta}} = \operatorname{Im} e^{2i\theta} \phi(re^{i\theta})$. With this notation, our integral becomes

$$\begin{aligned} & \lim_{L \rightarrow \infty} I_2 \\ &= \int_S \int_0^{2\pi} \frac{\operatorname{Im} e^{2i\theta} \phi(0)}{g_0(p)} e^{-t} \frac{\operatorname{Im} e^{2i\theta} \phi(r(t)e^{i\theta})}{g_0(p)(1-r(t)^2)^{-2}} \frac{dt d\theta d \operatorname{Area}}{2\pi \operatorname{Area}(S)} \\ &= \int_S \int_0^\infty \frac{e^{-t}(1-r(t)^2)^2}{g_0^2(p)} \int_0^{2\pi} \operatorname{Im}(e^{2i\theta} \phi(0)) \operatorname{Im}(e^{2i\theta} \phi(r(t)e^{i\theta})) \frac{dt d\theta d \operatorname{Area}}{2\pi \operatorname{Area}(S)}. \end{aligned}$$

Using the change-of-coordinates formula $e^{-t} dt = 2(1+r)^{-2} dr$ and the mean value theorem for harmonic functions (together with a half-angle formula to simplify the averaging), we find

$$\begin{aligned} \lim_{L \rightarrow \infty} I_2 &= \int_S \frac{1}{g_0(p)^2} \left(\int_0^1 2(1-r)^2 dr \right) \frac{|\phi(0)|^2}{2} \frac{d \operatorname{Area}}{\operatorname{Area}(S)} \\ (9.2) \quad &= \frac{1}{3} \int_S \frac{|\Phi(p)|^2}{g_0(p)^2} \frac{d \operatorname{Area}(p)}{\operatorname{Area}(S)}. \end{aligned}$$

Combining (9.1) and (9.2), we verify Theorem 9.1. In particular,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\text{Hess}\ell(\gamma_n)}{\ell(\gamma_n)} &= \frac{4}{3} \int_S \frac{|\Phi(p)|^2}{g_0(p)^2} \frac{d \text{Area}(p)}{\text{Area}(S)} \\ &= \frac{4}{3 \text{Area}(S)} \|\mu\|_{\text{WP}}^2. \end{aligned}$$

Remark. The constant $\frac{4}{3}$ found here agrees with that found by Wolpert [Wol86b] and McMullen [McM08]. See the comments ([McM08], p.376) of McMullen on the consistency of conventions.

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