HARMONIC EXTENSIONS OF QUASICONFORMAL MAPS TO HYPERBOLIC SPACE

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ABSTRACT. Abstract: We show that the set of quasiconformal (quasisymmetric, if $n = 2$) maps $h : S^{n-1} \to S^{n-1}$ which admit a quasi-isometric harmonic extension $H : \mathbf{H}^n \to \mathbf{H}^n$ is open in the set of quasiconformal (quasisymmetric, resp.) self-maps of S^{n-1}. The proof involves first deforming a harmonic map by a quasi-isometry, and then using that deformed map to set harmonic map Dirichlet problems on a compact exhaustion of $\mathbf{H}^n$. The solutions to these Dirichlet problems then converge to a harmonic map of bounded energy density which is at finite distance from the original deformed map.

§1. Introduction.

The goal of this note is to prove

Theorem A. *The set of quasiconformal (quasisymmetric, if $n = 2$) maps $h : S^{n-1} \to S^{n-1}$ which admit a quasi-isometric harmonic extension $H : \mathbf{H}^n \to \mathbf{H}^n$ is open in the set of quasiconformal (quasisymmetric, resp.) self-maps of S^{n-1}.*

As the set contains the identity, it is obviously non-empty. It seems to be more difficult to prove that this set is also closed; we hope to return to this in a later article.

There is a long history of research on the problem of finding nice extensions to $\mathbf{H}^n$ of quasiconformal (or quasisymmetric) homeomorphisms of S^{n-1}. Quasiconformal extensions were first constructed by Beurling and Ahlfors [BA] in dimension $n = 2$; higher dimensional extensions were given by Tukia and Väisälä [TV], and Tukia [T] produced a version that was compatible with the action of a group of Mobius transformations. Douady and Earle [DE] constructed a conformally natural version in all dimensions, and this led H.L. Royden (in 1985) to propose to the second author the problem of finding a harmonic extension, which might then also enjoy compatibility with Mobius transformations. These harmonic extensions were first constructed by Li and Tam ([LT1], [LT2]) under some assumptions

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of smoothness of the boundary maps $h : S^{n-1} \to S^{n-1}$ and a lower bound on its energy density. (See also Akutagawa [A].) Non-uniqueness properties of these extensions were also identified by Li-Tam ([LT2], [LT3]), which showed that the group compatibility properties would require some care; earlier, the second author had found similar non-uniqueness phenomena for some infinite energy harmonic maps of hyperbolic surfaces of finite volume [Wo]. Another approach for dimension $n = 2$ of studying the parametrization of harmonic self-maps of $\mathbf{H}^2$ via their Hopf differentials has been undertaken by Wan [Wa] and Tam-Wan [TW].

Theorem A is a partial result in the direction of the conjecture that all quasiconformal maps are harmonically extendible, and we found this theorem several years ago. Last year, Deane Yang independently obtained the same result, which he presents in [Y]. A case of this theorem, valid for maps h near the identity map in dimension $n = 2$, was proved a number of years ago by Earle and Fowler ([EF]) using implicit function theorem methods.

Our proof is rather straightforward. Given a biLipschitz harmonic map $H_0 : \mathbf{H}^n \to \mathbf{H}^n$, which thus has bounded energy density and quasiconformal boundary values $h_0 : S^{n-1} \to S^{n-1}$, we obtain the desired neighborhood of h_0 by considering maps $h : S^{n-1} \to S^{n-1}$ of the form $h = g \circ h_0$ where g is a quasi-conformal map of the sphere of small dilation δ. Choosing first a suitable *quasi-isometric* extension G of g, obtained by modifying [BA], [DE], or [TV], we then use the composition $G \circ H_0 : \mathbf{H}^n \to \mathbf{H}^n$ to set a series of boundary values $G \circ H_0 \big|_{\partial K_m}$ for a family of Dirichlet problems on compacta $\{K_m\}$ which exhaust $\mathbf{H}^n$. We estimate the distance from $G \circ H_0$ to the unique harmonic map $H_m : K_m \to \mathbf{H}^n$ whose boundary values agree with $G \circ H_0 \big|_{\partial K_m}$, and obtain a bound that is independent of m large and δ small. Uniform energy estimates then follow from Cheng's lemma [C], and we obtain our required map as a limit with $m \to \infty$.

Acknowledgments. We always benefit from our conversations with Cliff Earle, and this time was no exception.

§2. Background and Notation.

2.1. Harmonic Maps. Given a map $u : (M^m, g) \to (N^n, k)$ between Riemannian manifolds (M, g) and (N, k), the *energy* of u on a compact subset K of M is

$$\frac{1}{2} \int_K |\nabla u|^2 \; d\,\mathrm{vol}(g) \ ,$$

where, in local coordinates

$$|\nabla u|^2 = \sum g^{ij} \frac{\partial u^\alpha}{\partial x^i} \frac{\partial u^\beta}{\partial x^j} k_{\alpha\beta},$$
$$d\,\mathrm{vol}(g) = (\det g_{ij})^{1/2} dx \ .$$

The Euler-Lagrange equation for this energy functional is the condition for the vanishing of the *tension*, which is, in local coordinates,

$$\tau(u) = \Delta u^\gamma + {}^N\Gamma^\gamma_{\alpha\beta} u^\alpha_i u^\beta_j = 0 \ .$$

Solutions of $\tau(u) = 0$ are called *harmonic maps.*

Eells-Sampson [ES] showed that every homotopy class of maps from a Riemannian manifold M to a negatively curved target Riemannian manifold N contained a harmonic map, and Hamilton [Ham] extended that result to manifolds with boundary in a complete space.

2.2. The Riemannian Chain Rule and Distance in Hyperbolic Space. Let $u : (M, g) \to (N, k)$ be a C^2 map, not necessarily harmonic and $f : N \to \mathbf{R}$ be a function on N. Then we compute that, relative to an orthonormal frame $\{e_\alpha\}$ on M, we have

$$\Delta(f \circ u) = \mathrm{tr}_{\{u_* e_\alpha\}} \operatorname{Hess} f + \langle \operatorname{grad} f, \tau(u) \rangle_N . \tag{2.1}$$

Here, the first term on the right hand side is the trace with respect to the push-forward frame $\{u_* e_\alpha\}$ of the Riemannian Hessian, and the second term is the inner product of the gradient of f with the tension (vector) field $\tau(u)$.

A natural function to study on $\mathbf{H}^n$ is $\cosh d(\cdot, \cdot)$, as its Hessian has a particularly simple form. (Here $d(\cdot, \cdot)$ is the distance function on $\mathbf{H}^n$.) Given two points p, $q \in \mathbf{H}^n$, there is a unique geodesic γ_{pq} between them; we consider a frame in $\mathbf{H}^n$ adapted to γ_{pq} at p (at q, resp.) with one vector $\mathrm{Tan}_p \in T_p\mathbf{H}^n$ tangent along γ_{pq} ($\mathrm{Tan}_q \in T_q\mathbf{H}^n$ tangent along γ_{pq}) and normal vectors $N_1^p, \ldots, N_{n-1}^p \in T_p\mathbf{H}^n$ ($N_1^q, \ldots, N_{n-1}^q \in T_q\mathbf{H}^n$, resp.) With respect to the basis $(\mathrm{Tan}_p, 0)$, $(N_1^p, 0), \ldots, (N_{n-1}, 0)$, $(0, \mathrm{Tan}_q)$, $(0, N_1^q), \ldots, (0, N_{n-1}^q)$ for $T_{(p,q)}(\mathbf{H}^n \times \mathbf{H}^n)$, we find that the Hessian

$$\operatorname{Hess}_{\mathbf{H}^n \times \mathbf{H}^n} \cosh d = (\cosh d) I_{2n \times 2n} + A$$

where A is a $2n \times 2n$ matrix whose only non-zero entries are -1 at $((N_i, 0), (0, N_i))$, and a $-\cosh d$ at $((Tan_p, 0), (0, Tan_q)$ (and their symmetric places). Applying this to (2.1), we easily conclude that if $u : M \to \mathbf{H}^n$ and $w : M \to \mathbf{H}^n$ are maps from a Riemannian manifold to $\mathbf{H}^n$, then the function

$$Q : \mathbf{H}^n \times \mathbf{H}^n \to \mathbf{R}, \quad Q(x) = \cosh d(u(x), w(x)) - 1$$

satisfies

$$\begin{aligned} \Delta Q(x) &\geq \cosh d \left((\left\langle \nabla u, \mathrm{Tan}_{u(x)} \right\rangle - \left\langle \nabla w, \mathrm{Tan}_{w(x)} \right\rangle)^2 \right) \\ &\quad + (\cosh d - 1) \sum_{i=1}^{n-1} \left\langle \nabla u, N_i^{u(x)} \right\rangle^2 + \left\langle \nabla w, N_i^{w(x)} \right\rangle^2 \\ &\quad + \langle \operatorname{grad} Q, (\tau(u), \tau(w)) \rangle_{(u,w)(x)} \end{aligned} \tag{2.2}$$

where $\left\langle \nabla u, \mathrm{Tan}_{u(x)} \right\rangle$ refers to the projection of $\sum_{i=1}^n du(e_i)$ onto $\mathrm{Tan}_{u(x)}$.

2.3. Cheng's lemma. S.-Y. Cheng proved a lemma [C] that allows us to estimate the energy density of a harmonic map $u : M \to N$ between complete Riemannian manifolds, where N has non-positive sectional curvature and is simply connected, in terms of the geometry of the image $u(B)$ a ball $B \subset M$. His lemma is very flexible, but we will only need a special corollary (both of the statement and the proof).

Lemma 2.1 [C]. *Let M, N be simply connected Riemannian manifolds with N complete and $M \subset M'$ complete. Suppose all the sectional curvatures κ_M, κ_N are both non-positive. Then at a point $x_0 \in M$ which is at distance at least 1 from ∂M, we have the estimate*

$$|\nabla u|^2(x_0) \leq C(\kappa_N, R)$$

where R is chosen so that $B_R(u(x_0)) \supset u(B_1(x_0))$.

§3. Harmonic Extensions Near the Identity. In this section, we show that we can extend quasiconformal maps $h : S^{n-1} \to S^{n-1}$ with small dilatation, i.e., maps near the identity. This is our model argument, and the proof of Theorem A is a close analogue.

Proposition 3.1. *([EF] for $n = 2$) There is a $\delta > 0$ so that if $h : S^{n-1} \to S^{n-1}$ is a quasiconformal map with dilatation less than δ, then h is extendible to a harmonic map $H : \mathbf{H}^n \to \mathbf{H}^n$.*

Remark: We are grateful to Cliff Earle for informing us of this result of Fowler and himself. Their proof is an implicit function theorem that does not appear to generalize as easily as the proof below.

Proof: The method of proof is straightforward, as explained in the Introduction. First we extend the map to a controlled but not necessarily harmonic map $G : \mathbf{H}^n \to \mathbf{H}^n$. We then use G to set a family of Dirichlet problems for harmonic maps on a compact exhaustion $\{K_m\}$ of $\mathbf{H}^n$, obtaining a family of harmonic maps $H_m : K_m \to \mathbf{H}^n$. We uniformly bound the distance between $G\big|_{K_m}$ and H_m, as well as the energy density of H_m. This is enough to force H_m to subconverge to a harmonic map with the boundary values h.

3.1. We begin by observing that there are many ways presently known to extend a quasiconformal map $h : S^{n-1} \to S^{n-1}$ with small dilatation to a quasi-isometry $F : \mathbf{H}^n \to \mathbf{H}^n$ with biLipschitz constant near unity: see for example, the papers of Beurling-Ahlfors [BA], (for $n = 2$), Douady-Earle [DE; the relevant Theorem 5 is attributed to Tukia], or Tukia-Väisälä [TV].

Choose one such extension $F : \mathbf{H}^n \to \mathbf{H}^n$ which satisfies, for any unit vector $v \in T^1\mathbf{H}^n$, the inequalities

$$| \; \|dF[v]\| - 1 \; | < \epsilon_1 \tag{3.1}$$

where $\epsilon_1 = \epsilon_1(\delta) \to 0$ as $\delta \to 0$.

We want an extension that is also C^2 with pointwise small tension. To obtain this, we may, for example, divide $\mathbf{H}^n$ into compact isometric n-dimensional blocks, as in a standard dyadic decomposition of the upper half space model with totally geodesic faces. For any one such block B we may, by (3.1), associate a hyperbolic isometry G_B so that, for all $b \in B$,

$$d(F(b), G_B(b)) \; + \; \|(dF)_b - (dG_B)_b\| \; < \; \epsilon_2 \tag{3.2}$$

where $\epsilon_2 = \epsilon_2(\delta) \to 0$ as $\delta \to 0$. It follows that adjacent blocks have associated isometries that are $2\epsilon_2$ close as in (3.2). On a fixed size η tubular neighborhood of the dyadic $n-1$ skeleton one may locally smoothly interpolate between the isometries associated with the blocks of the adjacent faces. More precisely, one first interpolates in a neighborhood of each $n-1$ cell, away from the $n-2$ skeleton, between the 2 isometries associated with the adjoining n cells. Here one may get the tension bounded by $\frac{\epsilon_3}{\eta}$ where $\epsilon_3 = \epsilon_3(\delta) \to 0$ as $\delta \to 0$. The mapping is now defined on the boundary of a tubular neighborhood of the $n-2$ skeleton. One then extends by interpolating to this tubular neighborhood of the $n-2$ skeleton, staying away from a neighborhood of the $n-3$ skeleton. Continuing, one eventually gets the smooth map $G : \mathbf{H}^n \to \mathbf{H}^n$ satisfying

$$\|\tau(G)\| < \epsilon_4 = C\epsilon_3/\eta^n \tag{3.3}$$

and, as before, for any unit $v \in T^1\mathbf{H}^n$, the inequalities

$$|\ \|dG[v]\| - 1\ | < \epsilon_4 \tag{3.4}$$

where $\epsilon_4 = \epsilon_4(\delta) \to 0$ as $\delta \to 0$. Also since the pointwise distance $d(F, G)$ is bounded, in fact $o(1)$ as $\delta \to 0$, G has the same asymptotic boundary values as F and is again an extension of h.

Next let K_m be a compact exhaustion of $\mathbf{H}^n$, for example by concentric balls of radius about a given point; as in the choice of G, the precise nature of K_m is unimportant. Let H_m be the unique harmonic map $H_m : K_m \to \mathbf{H}^n$ with $H_m \big|_{\partial K_m} = G \big|_{K_m}$: the existence of the map H_m is guaranteed by the fundamental work of Hamilton [Ham], while the uniqueness is due to Al'ber [Al] and Hartman [Har].

3.2. To obtain estimates on the harmonic maps H_m that are independent of m and δ, we consider the function $Q_m(x) = \cosh d_{\mathbf{H}^n}(H_m, G) - 1$. We find, using (2.2), that

$$\Delta Q_m \geq \min_{\substack{v \in T^1_x\mathbf{H}^n \\ dG(v) \perp \gamma_x}} \left| dG(v) \right|^2 Q_m - \left\langle \tau(G), \operatorname{grad} d(\cdot, H_m) \big|_{G(x)} \right\rangle_{G(x)q} \sinh d(H_m, G) \tag{3.5}$$

where γ_x is the geodesic joining $G(x)$ to $H_m(x)$ so that its initial tangent vector is $-\operatorname{grad} d(\cdot, H_m) \big|_{G(x)}$ and its terminal tangent vector is $\operatorname{grad} d(G(x), \cdot) \big|_{H_m(x)}$. For any $x \in K_m$, for δ sufficiently small, we deduce from (3.4) the estimate:

$$\min_{\substack{dG(v)q \in T^1_x\mathbf{H}^n \\ v \perp \gamma_x}} |dG(v)|^2 > 1 - \epsilon_4(\delta)\ . \tag{3.6}$$

As $H_m \big|_{K_m} = G \big|_{K_m}$, we find that if H_m does not coincide with G on K_m, we must have all maxima of Q_m on the interior of K_m. At any such maximum, we apply inequalities (3.3) and (3.6), along with the obvious equality $|\operatorname{grad} d(\cdot, H_m)| = 1$, to (3.5) to find

$$\begin{aligned} 0 \geq \Delta Q_m &\geq (1 - \epsilon_4)Q_m - \epsilon_4(\tanh d(H_m, G))(Q_m + 1) \\ &\geq (1 - 2\epsilon_4)Q - \epsilon_4 \end{aligned}$$

so that at a maximum of Q_m, hence at all points of K_m, we have

$$Q_m \leq \epsilon_4/(1 - 2\epsilon_4) = \mu. \tag{3.7}$$

We observe that (3.7) is independent of the domain K_m.

3.3. Next we bound the energy density of H_m, using the gradient estimate of Cheng [C], which we described in section 2.3. This estimate of Cheng requires bounds on the diameter of the image of balls of H_m. But, for instance, for a ball $B_1(x)$ of radius 1 about a point x, we have

$$H_m(B_1(x)) \subset N_\mu G(B_1(x))$$

by (3.7), and since G is a $1+\epsilon_4$ quasi-isometry we see that

$$\operatorname{diam}(H_m(B_1(x))) \leq (1+\mu)(1+\epsilon_4) = C_0(\delta).$$

The estimate of Cheng (Lemma 2.1) then provides that

$$|\nabla H_m| \leq C_1(\delta). \tag{3.8}$$

3.4. Conclusion of the proof of Proposition 3.1: The bounds (3.7) and (3.8) suffice to allow us to invoke Ascoli-Arzela to find a subsequence of H_m which converges uniformly on compacta to a harmonic map $H : \mathbf{H}^n \to \mathbf{H}^n$ which is still at bounded distance μ from G. Thus, since $d(H,G) < \mu$, we see that H has the same asymptotic boundary values $h : S^{n-1} \to S^{n-1}$ as G, as desired. $\square$

§4. Proof of Theorem A.

Theorem A is a direct corollary of the proof of Proposition 3.1. We observe that all we used about the identity map id : $\mathbf{H}^n \to \mathbf{H}^n$ was that it was harmonic and biLipschitz. In particular if H_0 is a biLipschitz harmonic map, and $G \circ H_0$ a composition of H_0 with a quasi-isometry $G = G(\delta)$ satisfying (3.3) and (3.4), then the proof of (3.7) hinges on finding δ sufficiently small that

$$\left| \min_{\substack{v \in T_x^1 \\ d(G\circ H_0)(v) \perp \gamma_x}} d(G \circ H_0)v \right| > \tau(G \circ H_0). \tag{4.1}$$

However, the right-hand side goes to zero with δ while the left-hand side is bounded below by $\frac{1}{2}|\min_{v\in T_x^1} d(H_0)v|$, for small ϵ_1. Thus, the analogue of (3.7) holds under the more general hypothesis of Theorem A, and the rest of the argument is unchanged but for constants. $\square$

§5. Extensions of the Method.

5.1. Mobius Group Compatible Extensions. One of the motivations for studying harmonic extensions of quasiconformal homeomorphisms $h : S^{n-1} \to S^{n-1}$ is the compatibility they have with Mobius group actions (see also [DE]).

As harmonic extensions of a given homeomorphism are not necessarily unique, (see [Wo; Remark, p. 518] and, in the present context [LT3], we need to slightly restrict our family of extensions. It is convenient to note that an application of formula (2.2) yields

Lemma 5.1. *If H_0, $H_1 : \mathbf{H}^n \to \mathbf{H}^n$ are harmonic extensions of the same quasiconformal (quasi-symmetric) boundary homeomorphism , and if $d(H_0(x), H_1(x)) < C$ for every $x \in \mathbf{H}^n$, then $H_0 = H_1$.*

Proof: Formula (2.2) shows that $\cosh d(H_0, H_1)$ is a bounded subharmonic function on $\mathbf{H}^n$; an easy application of Yau's maximum principle shows that this is impossible unless the right hand side of (2.2) vanishes identically. But this would require both H_0 and H_1 to map to the same geodesic in $\mathbf{H}^n$, at a constant separation distance. Of course, this contradicts the boundary map $h : S^{n-1} \to S^{n-1}$ being a homeomorphism, with $h(S^{n-1})$ being much larger than merely the two endpoints of a geodesic. $\square$

With this observation, we observe that any harmonic map $H : \mathbf{H}^n \to \mathbf{H}^n$ which is a bounded distance away from our original extension $F : \mathbf{H}^n \to \mathbf{H}^n$ is uniquely determined. In particular, if a Mobius transformation $g : S^{n-1} \to S^{n-1}$ is compatible with $h : S^{n-1} \to S^{n-1}$ (i.e., $h \circ g = g \circ h$), then upon extending g to be an isometry of $\mathbf{H}^n$, and observing that both $g \circ H$ and $H \circ g$ are harmonic, we see that

Proposition 5.2. *Let Γ be a group of Mobius transformations. If $h : S^{n-1} \to S^{n-1}$ is Γ-compatible and $F : \mathbf{H}^n \to \mathbf{H}^n$ an extension which is nearly Γ-compatible (in the sense that there is a $C > 0$ for which $d(g \circ F, F \circ g) < C$ for all $g \in \Gamma$), then any unique harmonic extension $H : \mathbf{H}^n \to \mathbf{H}^n$ (with $\sup d(H, F) < \infty$) is also Γ-compatible.*

Of course, we have shown that the set of these Γ-compatible harmonic extensions is open, which is partial progress towards a problem of Tukia [T] on extending all such Γ-compatible homeomorphisms $h : S^{n-1} \to S^{n-1}$ to Γ-compatible quasi-isometries.

5.2.. Finally, we remark that the only place in our argument where we used that we had *constant* negative sectional curvature was in the computation of the precise constants in $\mathrm{Hess}_{\mathbf{H}^n \times \mathbf{H}^n} \cosh d$. This precision was not used in our proof, so the argument easily covers the case of Hadamard manifolds, i.e. simply connected complete negatively curved Riemannian manifold with fixed negative upper and lower bounds on sectional curvatures.

References

[Al] Al'ber, S.I., *On n-dimensional problems in the calculus of variations in the large*, Sov. Math. Dokl **5**, 700–704 (1964).

[A] K. Akutagawa, *Harmonic Diffeomorphisms of the Hyperbolic Plane*, Trans. Amer. Math. Soc. **342** (1994), 325–342.

[BA] A. Beurling and L. Ahlfors, *The Boundary Correspondence for Quasiconformal Mappings*, Acta Math. **96** (1956), 125-142.

[C] S.-Y. Cheng, *Liouville Theorem for Harmonic Maps*, Proc. Symp. Pure Math. **36** (1980), 147–151.

[DE] A. Douady and C. Earle, *Conformally Natural Extension of Homeomorphisms of the Circle*, Acta Math. **157** (1986), 23-48.

[EF] C. J. Earle and S. Fowler, private communication.

[ES] Eells, J. and Sampson, J.H., *Harmonic Mappings of Riemannian Manifolds*, Amer. J. Math. **86**, 109–160 (1964).

[Ham] R.S. Hamilton, *Harmonic Maps of Manifolds with Boundary*, Lecture Notes in Math. **471** (1975), Springer-Verlag, Berlin.

[H] Hartman, P., *On Homotopic Harmonic Maps*, Canad. J. Math. **19**, 673–687 (1967).

[LT1] P. Li and L.-F. Tam, *Uniqueness and Regularity of Proper Harmonic Maps*, Ann. of Math. **137** (1993), 167–201.

[LT2] ———, *Uniqueness and Regularity of Proper Harmonic Maps II*, Indiana J. of Math. **42** (1993), 591–635.

[LT3] ———, *The Heat Equation and Harmonic Maps of Complete Manifolds*, Invent. Math. **105** (1991), 1–46.

[T] P. Tukia, *Quasiconformal Extension of a quasisymetric mappings Compatible with a Mobius Group*, Acta math. **154** (1985), 153–193.

[TV] P. Tukia and J. Väisälä, *Quasiconformal extension from dimension n to $n+1$*, Ann. of Math. **115** (1982), 331–348.

[TW] L.-F. Tam and T.Y.-H. Wan, *Harmonic Diffeomorphisms Between Cartan-Hadamard Surfaces with Prescribed Hopf Differentials*, J. Differential Geom. **42** (1995), 368–410.

[Wa] T. Y.-H. Wan, *Constant Mean Curvature Surface, Harmonic Maps, and Universal Teichmuller Space*, J. Differential Geom. **35** (1992), 643-657.

[Wo] M. Wolf, *Infinite Energy Harmonic Maps and Degeneration of Hyperbolic Surfaces in Moduli Space*, J. Differential Geom. **33** (1991), 487–539.

[Y] D. Yang, *Deforming a Map into a Harmonic Map*, preprint.