The Weil-Petersson Isometry Group

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1 Introduction

Let $F = F_{g,n}$ be a surface of genus g with n punctures. We assume $3g-3+n > 1$ and that $(g,n) \neq (1,2)$. The purpose of this paper is to prove, for the Weil-Petersson metric on Teichmuller space $T_{g,n}$, the analogue of Royden's famous result [15] that every complex analytic isometry of $T_{g,0}$ with respect to the Teichmuller metric is induced by an element of the mapping class group. His proof involved a study of the local geometry of the cotangent bundle to Teichmuller space. Royden's result was extended to general $T_{g,n}$ by Earle-Kra [8], without any smoothness assumption on the isometry and with a stronger local result. They showed that if $2g+n > 4$ and $2g'+n' > 4$, and if f is an isometry from an open set $U \subset T_{g,n}$ to $T_{g',n'}$, then $T_{g,n} = T_{g',n'}$ and f is the restriction of an isometry induced by an element of the extended mapping class group. Later Ivanov [9] gave an alternative proof of Royden's theorem based upon the asymptotic geometry of Teichmuller space and the result that the group of automorphisms of the curve complex $C(F)$ (see below) coincides with the mapping class group. The automorphism result was later extended to the cases of punctured surfaces of genus $g \leq 1$ (with $(g,n) \neq (1,2)$) by Korkmaz [10], and at the same time proved for general $(g,n) \neq (1,2)$ by Luo [11].

We prove

Theorem A. *For $3g-3+n > 1$ and $(g,n) \neq (1,2)$, every Weil-Petersson isometry of Teichmüller space $T_{g,n}$ is induced by an element of the extended mapping class group $Mod^*(g,n)$.*

Our proof of this result is modelled somewhat on Ivanov's proof. We outline the ideas here. It is well-known that the Weil-Petersson metric is not complete ([22] and [7]). To complete the metric one adds a frontier $A_{g,n}$ to $T_{g,n}$; this frontier consists of a union of lower dimensional Teichmuller spaces. Each such space consists of Riemann surfaces with nodes or punctures. These surfaces are obtained by pinching nontrivial curves of F. Each Teichmuller space on the frontier carries its own Weil-Petersson metric and with this Weil-Petersson metric, this Teichmuller space on the frontier is isometrically embedded in the completion. We extend the isometry to the completion and show that each

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Teichmuller space on the frontier is preserved by the isometry. This self-map of the frontier then induces an automorphism of the complex of curves, $C(F)$. By Ivanov's result (and the extensions [10] [11] by Korkmaz and Luo to low genus) the automorphism is induced by an element of the extended mapping class group. (Ivanov's theorem is known not to hold for $(g, n) = (1, 2)$: see Luo [11]. Luo showed that $C(F_{1,2})$ is isomorphic to $C(F_{0,5})$; the automorphism group of this curve complex is the extended mapping class group $Mod^*(0, 5)$ and yet $Mod^*(1, 2)$ is a subgroup of index 5 in $Mod^*(0, 5)$).

Thus our isometry induces an extended mapping class group element at "infinity". We then show that the given isometry and the isometry induced by the corresponding element of the extended mapping class group act identically on Teichmuller space. The main tool, as in Ivanov, is to study geodesics in the space. In particular, after some formalities, we find that it is enough to assume that the isometry acts as the identity on the frontier $A_{g,n}$, and we study the (totally geodesic submanifold) Fix which is fixed by the isometry. General facts about fixed-point (proper) subsets of isometries in CAT(0) spaces yield points in $T_{g,n}$ at arbitrary distance from Fix. Yet, since Teichmüller space is a space of Riemann surfaces, additional estimates come from considering the functions $l_c(x) : T_{g,n} \to \mathbb{R}$, defined as the hyperbolic length of the curve c on the hyperbolic surface $x \in T_{g,n}$. A very strong such estimate is due to Wolpert [25], who shows that such functions are convex along Weil-Petersson geodesics. This allows us, following Wolpert, to give a center-of-mass argument that shows that Fix is non-empty. Combining this with the proof of Wolpert's result [23] that the Weil-Petersson metric is not complete, we find that Fix, along with all points in $T_{g,n}$, is uniformly close to a special set of frontier points. This contradiction with the general fact above about fixed-point proper subsets in CAT(0) spaces shows that Fix is all of $T_{g,n}$, proving the theorem.

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1.1 Teichmuller space, mapping class group and the Weil-Petersson metric

Denote by $\mathcal{M}$ the set of all smooth Riemannian metrics on F. Choose an orientation for F, and define the set of all similarly oriented complete hyperbolic structures on F by $\mathcal{M}_{-1}$: here $\mathcal{M}_{-1}$ naturally includes in $\mathcal{M}$. By the uniformization theorem, $\mathcal{M}_{-1}$ can be identified with the set of all conformal structures on F, with the given orientation. Equivalently, this is the same as the set of all complex structures or Riemann surface structures on F with the given orientation. The group of orientation preserving diffeomorphisms $Diff^+(F)$ acts on $\mathcal{M}_{-1}$ by pull-back. Let $Diff_0(F)$ the subgroup of diffeomorphisms isotopic to the identity. The Teichmuller space $T_{g,n}$ is defined to be

$$T_{g,n} = \mathcal{M}_{-1}/Diff_0(F).$$

We can equivalently define $T_{g,n}$ by fixing a complex structure S_0 on F and defining $T_{g,n}$ as the set of equivalence classes of pairs (S, f) where $f : S_0 \to S$ is a sense-preserving quasiconformal map from S_0 to S. Two pairs (S, f) and (S', f') are equivalent if there is a conformal map $h : S \to S'$ such that $h \circ f$ is homotopic to f'.

The mapping class group $Mod(g, n)$ is defined to be

$$Mod(g, n) = Diff^+(F)/Diff_0(F)$$

and the extended mapping class group is defined by

$$Mod^*(g, n) = Diff(F))/Diff_0(F).$$

The group $Mod^*(g, n)$ acts on $T_{g,n}$ as follows. We may choose S_0 so that it admits an antiholomorphic reflection $j : S_0 \to S_0$. Let $\Psi \in Mod(g, n)$ be represented by $\psi : S_0 \to S_0$. For $\psi : S_0 \to S_0$ orientation preserving, $\psi \cdot (S, f) = (S, f \circ \psi^{-1})$. Any orientation reversing diffeomorphism of S_0 can be expressed as $\psi \circ j$ for some orientation preserving ψ. Then $(\psi \circ j) \cdot (S, f)$ is the point $(S^*, f \circ j \circ \psi^{-1})$, where S^* is the conjugate Riemann surface to S; that is, the coordinate charts of S^* are those of S followed by complex conjugation.

The Moduli space $\mathcal{M}_{g,n}$ is defined to be $\mathcal{M}_{g,n} = T_{g,n}/Mod_{g,n}$.

It is well-known (see [14]) that $T_{g,n}$ has a complex structure. The cotangent space at a point $X \in T_{g,n}$ is the space of holomorphic quadratic differentials $\phi(z)dz^2$ on the Riemann surface X. On X there is a pairing of quadratic differentials and Beltrami differentials $\mu(z)\frac{d\overline{z}}{dz}$ (i.e. tensors of type $(-1, 1)$) on X. The pairing is given by

$$< \mu, \phi >= Re \int_X \mu(z)\phi(z) dz \wedge d\overline{z}.$$

Infinitesimally trivial Beltrami differentials μ are ones such that

$$< \mu, \phi >= 0$$

for all holomorphic ϕ. The tangent space at X is the space of Beltrami differentials modulo the infinitesimally trivial ones (see [3]).

The Weil-Petersson co-metric on $T_{g,n}$ is defined by the L^2-product on the cotangent bundle

$$< \phi_1(z)dz^2, \phi_2(z)dz^2 >= \frac{\sqrt{-1}}{2} \int_\Sigma \frac{\phi_1(z)\overline{\phi}_2(z)}{\lambda(z)} dz \wedge d\overline{z},$$

where $\lambda(z)|dz|^2$ is the hyperbolic metric on F. The metric is then defined on the tangent space by duality; alternatively, the metric is induced from the natural inner product on the tangent bundle $T\mathcal{M}$ to $\mathcal{M}$ along $\mathcal{M}_{-1}$, after projecting to $T_{g,n}$ (see [21]). The metric is Kahler [3]. The major properties that we will use are that the metric has negative sectional curvature ([16], [20], [24]) and Wolpert's remarkable result [25] that even though the metric is not complete, (see below) it is geodesically convex: there is a geodesic joining any two points, unique because of the negative curvature. We also note that the action of $Mod^*(g,n)$ on $T_{g,n}$ is isometric with respect to the Weil-Petersson metric.

1.2 The complex of curves, Ivanov's theorem, and the frontier of Teichmuller space

We define a complex $C(F)$ as follows. The vertices of $C(F)$ are homotopy classes of homotopically nontrivial, nonperipheral simple closed curves on F. An edge of $C(F)$ consists of a pair of homotopy classes of disjoint simple closed curves. More generally, a k-simplex consists of $k+1$ homotopy classes of mutually disjoint simple closed curves. The maximal number of mutually disjoint simple closed curves is $3g-3+n$ so that $C(F)$ is a $3g-4+n$ dimensional simplicial complex. The extended mapping class group $Mod^*(g,n)$ clearly acts on $C(F)$ by simplicial automorphisms. Namely for $\psi \in Mod^*(g,n)$ and $v_k = \{\beta_1, \ldots, \beta_{k+1}\}$ a k-simplex, the image $\psi(v_k)$ is the k-simplex $\{\psi(\beta_1), \ldots, \psi(\beta_{k+1})\}$. Ivanov proved [9] that in all but a few low genus cases, every simplicial automorphism of $C(F)$ is induced by some $g \in Mod^*(g,n)$ in the extended mapping class group. This was later extended by Korkmaz [10] to all cases except $(g,n) = (1,2)$. At the same time, Luo [11] gave an independent proof of all cases $(g,n) \neq (1,2)$ (with the explanation of the case $(g,n) = (1,2)$).

It is well-known that $T_{g,n}$ is not compact. For a simple closed curve β we can define a function $l_\beta : T_{g,n} \to R$ by setting $l_\beta(x)$ to be the length, in the hyperbolic metric on x, of the geodesic in the homotopy class of β. For a collection of curves $C = \beta_1, \ldots, \beta_N$, let

$$l_C = \sum_{i=1}^{N} l_{\beta_i}$$

One way to leave all compacta in $T_{g,n}$ is to choose a simplex v_k in $C(F)$, that is a set of disjoint simple closed curves $\beta_1, \ldots, \beta_{k+1}$, and form a sequence of Riemann surfaces along which the l_{β_i} go to 0. This motivates the definition of the *augmented Teichmuller space* ([5],[1]).

Specifically, associate to v_k the (possibly disconnected) surface $S\backslash\{\beta_1, \ldots, \beta_{k+1}\}$ whose components are punctured surfaces $S_1, \ldots, S_p$. Each S_i has its own Teichmuller space $T(S_i)$ and we let $\mathcal{O}(v_k)$ be the product $T(S_1) \times \ldots \times T(S_p)$ of Teichmuller spaces $T(S_i)$.

An alternative description is given in terms of surfaces with nodes (i.e. complex spaces in which each point has a neighborhood homeomorphic to either $\{|z| < \epsilon\}$ (regular points) or $\{(z,w) \in \mathbb{C}^2 | zw = 0, |z| < \epsilon, |w| < \epsilon\}$ (the *nodes*)). We identify each of those curves β_i to a point; the resulting space, say $S(v_k)$, is homeomorphic to a surface with nodes and we set $\mathcal{O}(v_k) = T(S(v_k))$. Here $T(S(v_k))$ is defined to be the product Teichmüller space of the punctured surfaces obtained by removing the nodal points from $S(v_k)$.

Thus, for each simplex v_k in $C(F)$ we consider a frontier Teichmuller space $\mathcal{O}(v_k)$ which is of complex dimension $3g-4+n-k$. We denote by $A_{g,n}$ the union of all these frontier Teichmuller spaces at infinity and by $\overline{T}_{g,n} = T_{g,n} \cup A_{g,n}$. There is a standard way (see [5],[1]) to topologize this union, sometimes referred to as the augmented Teichmüller space.

In this topology $\overline{T}_{g,n}$ is not compact, nor is it locally compact near a frontier point in $\mathcal{O}(v_k)$: take $\beta \in v_k$, and let τ be the Dehn twist about β. Then the τ orbit of a point $x \in T_{g,n}$ does not have a convergent subsequence.

By way of contrast, the action of the mapping class group extends to an action on $\overline{T}_{g,n}$ and the quotient is a compactification $\overline{\mathcal{M}}_{g,n}$ of the moduli space $\mathcal{M}_{g,n}$, commonly called the *Deligne-Mumford compactification* (see [2], [14]).

We may think of $\overline{T}_{g,n}$ as a stratified space, because if v_l is a subsimplex of v_k, then the Teichmuller space $\mathcal{O}(v_k)$ is on the frontier of the Teichmuller space $\mathcal{O}(v_l)$. This is because $k-l$ curves of the surfaces in $\mathcal{O}(v_l)$ have lengths that have become 0, or equivalently, the surfaces have acquired an additional $k-l$ nodes. All of these spaces lie in $A_{g,n}$, the union of the frontier spaces of $T_{g,n}$. For each $\mathcal{O}(v_k)$ we will denote by $Fr(\mathcal{O}(v_k))$ the union of its frontier spaces. Now the frontier $A_{g,n}$ is connected, although if we fix k, then each $\mathcal{O}(v_k)$ is a component of the union over all frontier Teichmuller spaces of that dimension.

The case of a maximal simplex v_{3g-4+n} is especially important, for in that case the resulting surface with punctures is a union of thrice-punctured spheres. Since the conformal structure (or equivalently, hyperbolic structure) on a three times punctured sphere is unique, the corresponding frontier Teichmuller space $\mathcal{O}(v_{3g-4+n})$ are singletons. We call these maximally pinched frontier spaces. They will play a crucial role in the sequel.

1.3 Incompleteness of Weil-Petersson metric and extension of isometries

Wolpert [22] and Chu [7] proved that the Weil-Petersson metric is not complete on $T_{g,n}$. In fact they showed that the Weil-Petersson distance in Teichmuller space to any frontier space $\mathcal{O}(v_k)$ is finite. Thus we can complete the metric by adding $A_{g,n}$, inducing a metric $dist_{v_k}$ on each frontier Teichmuller space $\mathcal{O}(v_k)$. Of course, each frontier Teichmuller space $\mathcal{O}(v_k)$ already has its own Weil-Petersson metric, written $d_{v_k}(\cdot,\cdot)$.

Lemma 1.3.1. *For two points p_0, p_1 in the same frontier space $\mathcal{O}(v_k)$, we have $d_{v_k}(p_0,p_1) = dist_{v_k}(p_0,p_1)$.*

Proof. The Weil-Petersson metric tensor in $T_{g,n}$ extends continuously to the Weil-Petersson metric tensor in $\mathcal{O}(v_k)$. ([12]) This implies that $d_{v_k}(p_0,p_1) \geq dist_{v_k}(p_0,p_1)$. On the other hand, suppose a length-minimizing path in the completion metric joining p_0 and p_1 enters $T_{g,n}$, and is thus a Weil-Petersson geodesic σ there. Without loss of generality we can assume σ lies inside $T_{g,n}$ except for its endpoints. Then l_{v_k} tends to zero near its endpoints, but is positive somewhere in its interior. This contradicts Wolpert's convexity result [25] which says that the functions l_β are strictly convex along Weil- Petersson geodesics. Thus $d_{v_k}(p_0,p_1) \le dist_{v_k}(p_0,p_1)$, completing the proof. $\square$

We will refer to $d_{WP}(\cdot,\cdot)$ as the completed metric on $\overline{T}_{g,n}$. We emphasize that the restriction of d_{WP} to any space $\mathcal{O}(v_k)$ is the Weil-Petersson metric on $\mathcal{O}(v_k)$.

Remarks. (i) Slightly stronger conclusions may also be drawn about this situation with additional use of Wolpert's convexity result ([25]). In particular, we see that each frontier space $\mathcal{O}(v_k)$, with the metric d_{v_k}, is geodesically embedded in $\overline{T}_{g,n}$, in the sense that any geodesic connecting a pair of points in $\mathcal{O}(v_k)$ lies entirely in that component $\mathcal{O}(v_k)$. To see this fact (to our knowledge, first written down in [26]), note first that such a geodesic, say σ, cannot meet a higher dimensional component $\mathcal{O}(v_{k-j})$, for $v_{k-j} \subset v_k$; as above, this is because the length of any curve in $v_k \setminus v_{k-j}$ is a non-negative convex function which vanishes at its endpoints, hence vanishes identically. On the other hand, since $\mathcal{O}(v_k)$ is a product of Teichmuller spaces (with the Weil-Petersson metric) of punctured surfaces, we see that on the geodesic σ, the length functions are bounded by the maximum of their values at their endpoints, and hence σ meets no frontier spaces $\mathcal{O}(v_{k+l})$ on the frontier of $\mathcal{O}(v_k)$.

(ii) We note that since the Weil-Petersson metric has negative curvature and is geodesic convex, $T_{g,n}$ is a $CAT(0)$ space with this metric. B. Farb points out the general fact ([6], Corollary II.3.11) that the metric completion of a geodesically convex $CAT(0)$ space is $CAT(0)$. Thus in particular $\overline{T}_{g,n}$ is $CAT(0)$ and thus between any two points there is a unique geodesic. We however will not need to use this last fact.

We will need the following lemmas.

Lemma 1.3.2. *Fix a frontier space $\mathcal{O}(v_k)$ (which may be $T_{g,n}$ itself). Then, for all ϵ sufficiently small, there exists $L = L(\epsilon)$, depending only on ϵ with the following property. If $\mathcal{O}(v_{3g-4+n})$ is any maximally pinched frontier point on the frontier of $\mathcal{O}(v_k)$, and C is the collection of curves in $v_{3g-4+n} \setminus v_k$, then if $d_{WP}(x, \mathcal{O}(v_{3g-4+n})) \le \epsilon$, we have $l_C(x) \le L$.*

Proof. If not, there exists a sequence $\{x_j\} \in \mathcal{O}(v_k)$ and a sequence $\{\mathcal{O}(v_{3g-4+n,j})\}$ of maximally pinched frontier points on the frontier of $\mathcal{O}(v_k)$ with $d_{WP}(x_j, \mathcal{O}(v_{3g-4+n,j})) \to 0$ and $l_{C_j}(x_j) \to \infty$; here, C_j of course refers to the curves $v_{3g-4+n,j} \setminus v_k$. Since there are but a finite number of homotopy classes of maximally pinched surfaces, we can find subsequences, again called $\{x_j\}$ and $\{\mathcal{O}(v_{3g-4+n,j})\}$ and a sequence $\{f_j\} \subset Mod(g,n)$ so that $f_j\mathcal{O}(v_{3g-4+n,j}) = \mathcal{O}(w_{3g-4+n})$, where $\mathcal{O}(w_{3g-4+n})$ is some single maximally pinched frontier point on the frontier of $\mathcal{O}(v_k)$. Then let C denote the curve system $C = w_{3g-4+n} \setminus v_k$, so that $C = f_jC_j$, and set $y_j = f_jx_j$. Then, since f_j induces a Weil-Petersson isometry, we have $d(y_j, \mathcal{O}(w_{3g-4+n})) = d(f_jx_j, f_j\mathcal{O}(v_{3g-4+n,j})) = d(x_j, \mathcal{O}(v_{3g-4+n,j})) \to 0$, while

$l_C(y_j) = l_{f_j C_j}(f_j x_j) = l_{C_j}(x_j) \to \infty$: this last equality follows from f_j amounting to but a consistent relabelling of curves and hyperbolic surfaces. But then the first limit implies that the sequence $\{y_j\}$ converges to $\mathcal{O}(w_{3g-4+n})$. However this implies $l_C(y_j) \to 0$ and we have a contradiction with the second limit statement. □

Lemma 1.3.3. *Fix a simplex $v_k \subset C(F)$. Given $\rho > 0$ and $M > 0$ there is a $\delta = \delta(\rho, M)$ with the following property. Let C be a collection of curves on the surface $S = F \setminus v_k$. If $x \in \mathcal{O}(v_k)$ satisfies $l_C(x) \leq M$ and $d_{WP}(x, Fr(\mathcal{O}(v_k)) \geq \rho$, then any y in the Weil-Petersson δ ball about x satisfies $l_C(y) \leq 2M$*

Here $Fr(\mathcal{O}(v_k)$ denotes the frontier of $\mathcal{O}(v_k)$.

Note there must be some condition on the location of x for the conclusion to hold. Take a frontier space $\mathcal{O}(w)$ of $\mathcal{O}(v_k)$ obtained by pinching along a curve β not in C, but which intersects some curve in C. In any δ neighborhood of a point in $\mathcal{O}(w)$ there are points $x, y \in \mathcal{O}(v_k)$ which differ by arbitrarily large powers of the Dehn twist about β. But then the ratio of $l_C(x)$ to $l_C(y)$ can be made arbitrarily large.

Proof. If the lemma were not true, there would be sequences $\{x_j\}, \{x'_j\} \in \mathcal{O}(v_k)$ and a sequence $\{C_j\}$ of curve families such that $d_{WP}(x_j, x'_j) \to 0$, $d_{WP}(x_j, Fr(\mathcal{O}(v_k)) \geq \rho$, and $l_{C_j}(x_j) \leq M$, while $l_{C_j}(x'_j) > 2M$. ¿From the first two conditions we can find subsequences again denoted $\{x_j\}, \{x'_j\}$ and a sequence $\{f_j\} \subset Mod(g, n)$ such that $\{y_j = f_j(x_j)\}$ and $\{y'_j = f_j(x'_j)\}$ converge to the same point $y_0 \in \mathcal{O}(v_k)$. Fix K a compact subset of $\mathcal{O}(v_k)$ containing y_j, y'_j for j large. Now $\{C'_j = f_j(C_j)\}$ is a sequence of curve families such that $l_{C'_j}(y_j) = l_{f_j(C_j)}(f_j(x_j)) = l_{C_j}(x_j) \leq M$, while $l_{C'_j}(y'_j) > 2M$. Since K is compact, there are only finitely many curves β such that $l_\beta(z) \leq M$ for some $z \in K$. Thus $\bigcup_j C'_j$ is actually a finite set of curves, and by passing to subsequences we can assume the sequence $\{C'_j\}$ is a fixed set C_0. But now the length functions $l_\beta(\cdot)$ are continuous on $\mathcal{O}(v_k)$ and we have contradicted the fact that both $\{y_j\}$ and $\{y'_j\}$ converge to y_0. □

Now suppose I is an isometry of $T_{g,n}$ into itself in the Weil-Petersson metric. Since we do not a priori assume that the isometry I is surjective, we first show

Lemma 1.3.4. *$I : T_{g,n} \to T_{g,n}$ is surjective.*

Proof. Since I is an open map, we need to show that I is proper. Suppose on the contrary that there is a sequence $\{x_j\}$ leaving every compact set in $T_{g,n}$ such that $\{I(x_j)\}$ lies in a compact set K of $T_{g,n}$. We first show that no subsequence of $\{x_j\}$ can project to a precompact open set M in the moduli space $\mathcal{M}_{g,n}$. For as the compact set K has finite Weil-Petersson diameter, so does $I^{-1}(K) \subset T_{g,n}$; since a set of finite diameter can intersect only finitely many disjoint balls of fixed diameter, $I^{-1}(K)$ could intersect but a finite number of preimages of the precompact open set $M \subset \mathcal{M}_{g,n}$ under the projection map $T_{g,n} \to \mathcal{M}_{g,n}$. Thus the subsequence $\{x_j = I^{-1}I(x_j)\}$ would be contained in the closure of the union of those finite number of precompact preimages, and hence would be contained in a compact set in $T_{g,n}$, contrary to hypothesis.

We conclude that the entire sequence projects to a sequence that leaves every compact subset of $\mathcal{M}_{g,n}$. However along such a sequence $\{x_j\}$ the infimum of the scalar curvature of the Weil-Petersson metric is $-\infty$ [19], while bounded in the compact set K. But this is a contradiction to the fact that an isometry preserves scalar curvature. Thus I is in fact surjective. $\square$

It is immediate that I is therefore also invertible.

We discuss next how I extends to an isometry $\overline{I} : \overline{T}_{g,n} \to \overline{T}_{g,n}$ and that this extension preserves each frontier space $\mathcal{O}(v_k)$.

Lemma 1.3.5. *The isometry I extends to an isometry $\overline{I} : \overline{T}_{g,n} \to \overline{T}_{g,n}$ of $\overline{T}_{g,n}$ which is surjective. For each $k = 0, \ldots, 3g - 4 + n$, the isometry $\overline{I}$ sends each frontier space $\mathcal{O}(v_k)$ determined by a k-simplex v_k onto a frontier space $\mathcal{O}(w_k)$, where w_k is another k-simplex.*

We have the immediate corollary.

Corollary 1.3.6. *I induces an automorphism $\widehat{I}$ of the curve complex $C(F)$.*

We prove Lemma 1.3.5. The statement that I extends is immediate, holding for any isometry of the completion of a metric space. Denote the extension again by I. The proof of Lemma 1.3.4 shows that the frontier $A_{g,n}$ must be mapped to itself. Since every point of $A_{g,n}$ is a limit of points of $T_{g,n}$ it follows immediately that the extension I must map $A_{g,n}$ isometrically onto itself.

Now we wish to show the second statement; that each point in $\mathcal{O}(v_k)$ is mapped to $\mathcal{O}(w_k)$ for some k-simplex w_k. We first show inductively that it is not possible for either I or I^{-1} to map a point in $\mathcal{O}(v_k)$ to a point in $\mathcal{O}(w_l)$ for a simplex w_l with $l < k$. Since $A_{g,n}$ is mapped to itself, the induction statement is true for $k = 0$. (Here $k = -1$ corresponds to $T_{g,n}$) Suppose the induction step for both I and I^{-1} is true for all $l < k$ but a point $x_0 \in \mathcal{O}(v_k)$ is mapped by I to $y_0 \in \mathcal{O}(w_l)$ for $l < k$. For some l-simplex v_l, the point x_0 is in the frontier of $\mathcal{O}(v_l)$. Let U be the intersection of a neighborhood of x_0 in $\overline{T_{g,n}}$ with $\mathcal{O}(v_l)$. By the induction hypothesis, the points in U must map into a neighborhood V of y_0 in $\mathcal{O}(w_l)$ (and not to to points in a higher dimensional space corresponding to a lower dimensional simplex).

But now I is an isometry of Weil-Petersson metrics from U to V; consider a sequence in U going to the frontier point x_0 whose image sequence converges to a point in V. This again contradicts the fact that the scalar curvature goes to $-\infty$ along the sequence in the domain and stays bounded on the image sequence. Thus the induction step holds for I. The argument for I^{-1} is identical. We conclude each point in $\mathcal{O}(v_k)$ goes to a point in $\mathcal{O}(w_k)$. Now since each $\mathcal{O}(v_k)$ is a connected component in the union (over all k-simplices w_k) of $\mathcal{O}(w_k)$, and I is a continuous map of $A_{g,n}$, we conclude that $\mathcal{O}(v_k)$ must map to a single $\mathcal{O}(w_k)$.

$\square$

2 Proof of Theorem A

By Corollary 1.3.6, the isometry I induces an automorphism $\widehat{I}$ of the curve complex $C(F)$. By Ivanov's theorem (as extended by Korkmaz and Luo), for $(g,n) \neq (1,2)$, the automorphism $\widehat{I}$ agrees with the automorphism of $C(F)$ induced by a mapping class, say $\varphi : F \to F$, which may be orientation reversing. As we have seen, the automorphism φ induces a Weil-Petersson isometry Φ of $T_{g,n}$; then $I \circ \Phi^{-1}$ is a Weil-Petersson isometry of $T_{g,n}$ extending to an isometry of $\overline{T}_{g,n}$ and preserving each frontier space $\mathcal{O}(k)$. Thus the isometry $I \circ \Phi^{-1}$ induces the identity on $C(F)$. At this point then we lose no generality in the argument while simplifying the notation if we assume that $I = \Phi$ preserves each $\mathcal{O}(v_k)$ (inducing the identity on $C(F)$) and we are seeking to prove that I is also the identity map on Teichmüller space. Roughly then, our goal is to move from rough knowledge of $I : \overline{T}_{g,n} \to \overline{T}_{g,n}$ on the frontier $A_{g,n}$ to precise control on $I : T_{g,n} \to T_{g,n}$ on the interior $T_{g,n}$ of $\overline{T}_{g,n}$. In fact we prove, by induction on the dimension of the frontier spaces, that I is the identity on each space $\mathcal{O}(v_k)$ and hence on $T_{g,n}$.

The induction hypothesis holds for the lowest dimensional frontier spaces, namely the maximally pinched frontier spaces corresponding to the maximal simplices in $C(F)$, since the maximally pinched frontier spaces are singletons and are fixed by I. Our induction hypothesis is then that for all $l > k$, the isometry I is the identity for all frontier Teichmuller spaces $\mathcal{O}(v_l)$ of dimension $3g-4+n-l$ and $\mathcal{O}(v_k)$ is a frontier space of dimension $3g-4+n-k$ (which is $T_{g,n}$ itself, if $k=-1$) Consider now I restricted to $\mathcal{O}(v_k)$. The space $\mathcal{O}(v_k) = T(S)$ is the Teichmuller space $T(S)$ of some surface S.

Let $Fix = Fix(\mathcal{O}(v_k))$ denote the fixed-point set of I acting on $\mathcal{O}(v_k)$. It is a general fact that the fixed-point set of an isometry of a Riemannian CAT(0) space is a totally geodesic submanifold. (To see this in our setting, note that Wolpert's result [25] says that there is a geodesic between any two points of $\mathcal{O}(v_k)$ and the negative sectional curvature of $\mathcal{O}(v_k)$ says that this geodesic is unique. Consequently, the geodesic joining any two points of Fix is also fixed by I and Fix is a convex subspace of $\mathcal{O}(v_k)$. Next consider the action of dI on the tangent space $T_zT(S)$ for $z \in Fix$. We see that dI fixes the initial tangent vector V_ζ to any geodesic from z to any other point in Fix. Thus a neighborhood of z in Fix is given by the exponentiated image of the kernel of dI. This shows that Fix is a submanifold. Thus Fix is a totally geodesic submanifold of $\mathcal{O}(v_k)$.)

Now suppose Fix is a proper subset of $\mathcal{O}(v_k)$. Let $N^1(Fix) \subset TT(S)$ be the unit normal bundle to Fix, thought of as a subbundle to the tangent bundle to $\mathcal{O}(v_k)$.

We postpone the proof that $Fix \neq \emptyset$, while we discuss properties of Fix that follow directly from $d_{WP}|_{\mathcal{O}(v_k)}$ being $CAT(0)$.

We say that a vector $v \in N^1_p(Fix)$ "exponentiates" to the frontier if the geodesic $\{exp_p sv | s \in [0, L]\}$ determined by v joins p to a frontier point $y = exp_p Lv$. We claim that in fact no vectors exponentiate to the frontier. For again by the induction hypothesis, if such a geodesic would exist, the isometry I would fix y and p and so would fix the geodesic. However, the geodesic is normal to Fix, hence not contained in Fix, and we have a contradiction.

Therefore let $v_0 \in N^1(Fix)$ be any vector tangent to say $p_0 \in Fix$, i.e. $v_0 \in T_{p_0}T(S)$. Let γ_0 the exponentiated image of v_0 and let $p_s = \exp_{p_0} sv_0$ be a point at distance s along γ_0. We claim that

Claim 2.0.7. $d_{WP}(p_s, Fix) = s$

This claim of course follows if we know that the distance from p_s to Fix is minimized at p_0. To see this, let $q \in Fix$ be any point in Fix other than p_0. The triple p_s, p_0, q of points form a right triangle in the CAT(0) space $T(S)$. (See [6], Chapter II.1 for a discussion of CAT(0) and Riemannian angles, and their equivalence in this case.) But then the distance $d_{WP}(q, p_s)$ is at least as large as the comparison Euclidean distance $d_{\mathbb{E}^2}(\overline{q}, \overline{p_s})$ (in the obvious notation) and $d_{\mathbb{E}^2}(\overline{q}, \overline{p_s}) > d_{\mathbb{E}^2}(\overline{p_0}, \overline{p_s}) = d_{WP}(p_0, p_s)$: here the inequality follows because $\overline{qp_s}$ is the hypotenuse of a Euclidean right triangle, and the equality follows by construction of the Euclidean comparison triangle. □

Note the claim says that there are points in $\mathcal{O}(v_k)$ arbitrarily far from Fix. However we will now also show that

Claim 2.0.8. *There is some M such that every point in $\mathcal{O}(v_k)$ is within distance M of Fix.*

This contradiction between the claims will then show that $Fix = \mathcal{O}(v_k)$, completing the proof of the induction step.

To show that every point of $\mathcal{O}(v_k)$ is within some distance M of Fix, we first note that every point of $\mathcal{O}(v_k)$ is within some universally bounded distance d_0 of *some* maximally pinched frontier point. That statement follows from two others. The first ([5], Theorem XV) is that for any hyperbolic surface there is a maximal set of disjoint curves with universally bounded hyperbolic lengths. Such a surface is then of universally bounded distance from the corresponding maximally pinched frontier point by the proof in [22]. (Specifically, by [18] there exists a holomorphic quadratic differential Ψ with closed horizontal trajectories homotopic to the elements of a maximal set of disjoint curves such that the corresponding cylinders have equal moduli. Now our upper bound for the hyperbolic lengths implies an upper bound for the extremal lengths; it is then easy to check that these conditions imply a lower bound on the moduli of those cylinders. Wolpert's proof in [22] then gives a *uniform* upper bound on the Weil-Petersson length of the Teichmuller geodesic, corresponding to Ψ and tending to the maximally pinched frontier point corresponding to the surface pinched along each of the specified curves in the maximal family.)

We are reduced to showing the following proposition.

Proposition 2.0.9. *There exists M' such that the M'-neighborhood of any maximally pinched frontier point $\mathcal{O}(v_{3g-4+n})$ of $\mathcal{O}(v_k)$ contains points of Fix.*

Remark. This claim would follow immediately, if we knew that the interiors of geodesics between maximally pinched frontier points lay in a single component of $\overline{\mathcal{O}(v_k)}$. For if so, then consider a maximally pinched frontier point $\mathcal{O}(v_{3g-4+n})$

of $\mathcal{O}(v_k)$, and choose another maximally pinched frontier point $\mathcal{O}(v'_{3g-4+n})$ of $\mathcal{O}(v_k)$ so that the curves in v_{3g-4+n} together with those in v'_{3g-4+n} fill the surface S. Then a geodesic between $\mathcal{O}(v_{3g-4+n})$ and $\mathcal{O}(v'_{3g-4+n})$ is in $\overline{\mathcal{O}(v_k)}$ as well as in Fix, the latter because geodesics in $CAT(0)$ spaces like $\overline{\mathcal{O}(v_k)}$ are unique. As the geodesic obviously limits on $\mathcal{O}(v_{3g-4+n})$, the Proposition follows.

Proof. For each $x \in \mathcal{O}(v_k)$ we are interested in its I orbit. Let

$$Orb(x) = \{I^j(x)\}_{j=-\infty}^{\infty}$$

Since I is an isometry and fixes the frontier of $\mathcal{O}(v_k)$ by the induction hypothesis, each point of $Orb(x)$ is the same distance from the frontier of $\mathcal{O}(v_k)$ as x is.

Fix a small $\rho > 0$ and let

$$\mathcal{O}(v_k, \rho) = \{x \in \mathcal{O}(v_k) : d_{WP}(x, Fr(\mathcal{O}(v_k))) > \rho\}$$

It is easy to see $\mathcal{O}(v_k, \rho)$ is open and connected. Let $C = C(v_{3g-4+n})$ denote the curves in $v_{3g-4+n} \setminus v_k$. Now let

$$\Omega(C, \rho) = \{x \in \mathcal{O}(v_k, \rho) : \sup_{y \in Orb(x)} l_C(y) < \infty\}$$

That is, $\Omega(C, \rho)$ consists of those points such that the lengths of the curves in C are bounded on the entire orbit.

We begin by claiming that

Claim 2.0.10. $\Omega(C, \rho) = \mathcal{O}(v_k, \rho)$.

We prove the claim by showing that $\Omega(C, \rho)$ is non-empty, open and closed in the connected set $\mathcal{O}(v_k, \rho)$.

We first show that $\Omega(C, \rho)$ is nonempty. Using the isometric action of the mapping class group there exists $\epsilon > 0$ and $\rho > 0$ and x_0 so that $d_{WP}(x_0, \mathcal{O}(v_{3g-4+n})) = \epsilon$ and $d_{WP}(x_0, Fr(\mathcal{O}(v_k)) = 2\rho$. We may choose ϵ (and ρ) small enough so that the hypothesis of Lemma 1.3.2 holds. Then since the orbit of I remains within ϵ of $\mathcal{O}(v_{3g-4+n})$, Lemma 1.3.2 says that $x_0 \in \Omega(C, \rho)$.

We now show that $\Omega = \Omega(C, \rho)$ is open. Let $x \in \Omega$ and let $L = L(x) = \sup_{y \in Orb(x)} l_C(y)$. By Lemma 1.3.3, there exists $\delta = \delta(\rho, L)$ such that for all $y \in Orb(x)$ and all ζ in the δ ball about y,

$$l_C(\zeta) \leq 2L$$

Since I is an isometry, the I orbit of a δ ball about x is the union of the δ balls about the points in $Orb(x)$. Since $\mathcal{O}(v_k, \rho)$ is open, if we take δ small enough we can insure the δ ball about x remains in $\mathcal{O}(v_k, \rho)$ and the inequality above says that it is then contained in Ω, proving Ω is open.

Finally, we show Ω is closed. Let z_0 be a limit of points $z_i \in \Omega$ and assume $z_0 \notin \Omega$. Then there exists a sequence $\{y_j = I^j(z_0)\} \subset Orb(z_0)$ such that

$l_C(y_j) \to \infty$. Since $y_j \in \mathcal{O}(v_k, \rho)$ there is a compact set $K \subset \mathcal{O}(v_k, \rho)$ and a sequence $\{f_j\} \subset Mod(g,n)$ such that $\zeta_j = f_j(y_j) \in K$. Let $C_j = f_j(C)$ be the collection of image curves. We have

$$l_{C_j}(\zeta_j) = l_{f_j(C)}(f_j(y_j)) = l_C(y_j) \to \infty$$

Since I is an isometry and f_j induces an isometry, we have that for i large enough, the points $\zeta_{j,i} = f_j(I^j(z_i))$ lie in K. Fix such an index i and consider the sequence $\{\zeta_{j,i}\}$. Now we have

$$\sup_j l_{C_j}(\zeta_{j,i}) < \infty$$

However, given a compact set K, there exists a constant L_0 (depending only on K) such that for any two points $x, y \in K$ and *any* curve β we have $\frac{l_\beta(x)}{l_\beta(y)} \leq L_0$. This contradicts the previous assertions that the ratio of $l_{C_j}(\zeta_j)$ and $l_{C_j}(\zeta_{j,i})$ have no bound; thus z_0 must in fact lie in Ω, and Ω is closed in $\mathcal{O}(v_k, \rho)$. This concludes the proof of the claim.

We now conclude the proof of the Proposition. Now let $\mathcal{O}(w_{3g-4+n})$ be another maximally pinched frontier point of $\mathcal{O}(v_k)$ such that the curves in $C = v_{3g-4+n} \setminus v_k$ together with those in $C' = w_{3g-4+n} \setminus v_k$ *fill* the surface S, which means that if we remove the curves in $C \cup C'$ from S, the result is a union of simply connected domains.

We form the corresponding $\Omega(C', \rho)$, and again the claim above shows that this coincides with $\mathcal{O}(v_k, \rho)$. Thus every point in $\mathcal{O}(v_k, \rho)$ lies in $\Omega(C, \rho) \cap \Omega(C', \rho)$.

Let x_0 be a point at distance 2ρ from the frontier and ϵ from $\mathcal{O}(v_{3g-4+n})$. Applying Lemma 1.3.3 again, there is a ball B of radius δ about x_0 such that if we set

$$B_0 = \cup_{j=-\infty}^{\infty} I^j(B)$$

then there exists M_1 such that $l_{C \cup C'}(x) < M_1$ for all $x \in B_0$. We may take $\delta < \rho$. Clearly B_0 is I invariant.

Now we adapt an argument of Wolpert's [25] in his proof of the Nielsen Realization Problem that every finite subgroup of the mapping class group has a fixed point to find a fixed point in the current situation.

Consider the subset

$$W = W(C \cup C') = \{x \in \mathcal{O}(v_k) : l_{C \cup C'}(x) \leq M_1\}$$

By [25], since the set of curves $C \cup C'$ fills S, the set W is a compact subset of $\mathcal{O}(v_k)$; also the set W is a cell, its boundary is C^1, and it contains B_0 in its interior.

Now define a function D on $\mathcal{O}(v_k)$ by

$$D(x) = \frac{1}{\mu(B_0)} \int_{B_0} d_{WP}(x, y) d\mu(y),$$

where μ is Weil-Petersson volume element. As B_0 is a subset of the compact set W, the set B_0 has finite total measure. Further, as $d_{WP}(x, \cdot)$ is bounded on $W \supset B_0$, we see that $D(x)$ is a finite integral for each x.

Since B_0 is I-invariant, so is D. Since $l_C(x) > l_C(y)$ for x in the boundary of W and $y \in B_0$, the strict convexity of $l_C(x)$ along Weil-Petersson geodesics says that the vector $\mathrm{grad}_{WP}\, d_{WP}(\cdot, y)$ points out at each point on the boundary of W. Thus as an average of such vectors, $\mathrm{grad}_{WP}\, D(\cdot)$ points out at each such boundary point. By the Poincare-Hopf index theorem, we see that there is some $y_0 \in W$ for which $\mathrm{grad}_{WP}\, D(y_0) = 0$. By the negative curvature and the geodesic convexity of the metric, the distance from a point to a geodesic is a strictly convex function of the parameter along the geodesic. Consequently as an average of such functions, D is also strictly convex. This implies that y_0 is the unique minimum for D_0. It follows that $I(y_0) = y_0$.

Now we wish to estimate $d_{WP}(y_0, \mathcal{O}(v_{3g-4+n}))$. Each point of B_0 is distance at most $\epsilon + \delta \leq \epsilon + \rho$ from $\mathcal{O}(v_{3g-4+n})$. Let $y_1 \in \mathcal{O}(v_k)$ be any point within distance ρ of $\mathcal{O}(v_{3g-4+n})$ and thus y_1 is at most $\epsilon + 2\rho$ from any point of B_0. Since D is the average of such distances, $D(y_1) \leq \epsilon + 2\rho$. Since y_0 is the point that minimizes D,

$$D(y_0) \leq D(y_1) \leq \epsilon + 2\rho$$

Since $D(y_0)$ is the average of distances frm y_0 to points in B_0,

$$\min_{y \in B_0} d_{WP}(y_0, y) \leq \epsilon + 2\rho.$$

Since we have the bound $\epsilon + \rho$ on the distance from any point of B_0 to $\mathcal{O}(v_{3g-4+n})$,

$$d_{WP}(y_0, \mathcal{O}(v_{3g-4+n})) \leq 2\epsilon + 3\rho$$

We have thus found a point of Fix within uniform distance $M' = 2\epsilon + 3\rho$ of every maximally pinched frontier of $\mathcal{O}(v_k)$. □

The proof of the main Theorem A is now complete.

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