UPPER BOUND APPROXIMATION
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Frequently in analysis we need to prove an inequality of the form

(1) \[ A \leq B \]

but, due to problems bounding \( A \) or \( B \), we are only able to establish that

(2) \[ A \leq B + \epsilon \]

for some small \( \epsilon > 0 \). However, if Equation 2 holds for all \( \epsilon > 0 \), then we can infer the stronger inequality in Equation 1!

Let’s break down the argument rigorously:

**Technique** (Upper Bound Approximation). Let \( A \leq B + \epsilon \) for all \( \epsilon > 0 \). Then \( A \leq B \).

**Proof.** We will prove this statement by contradiction. Therefore, we will assume that \( A > B \).

*Step 1:* There is a \( \delta > 0 \) such that \( A > B + \delta \). For example, let \( \delta = (A - B)/2 \).

*Step 2:* As given, \( A \leq B + \epsilon \) for all \( \epsilon \), so in particular \( A \leq B + \delta \).

We therefore have a contradiction, because \( A \leq B + \delta \) and \( A > B + \delta \) cannot both be true simultaneously! \( \square \)

It is important to verify that \( A \) and \( B \) do not depend on \( \epsilon \), or this conclusion will not hold in general.

With this argument under our belts, let’s look at an example. First, we recall the following property of supremums over \( \mathbb{R} \):

**Property** (\( \epsilon \) property of supremums). If \( A \subseteq \mathbb{R} \) is a nonempty set of real numbers that is bounded above, then for all \( \epsilon > 0 \), there is an \( a \in A \) such that \( \sup A \leq a + \epsilon \).

We can then prove the following proposition:

**Proposition.** If \( A, B \) are nonempty sets of positive real numbers, then \( \sup(A \cdot B) \leq \sup(A) \cdot \sup(B) \).

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Proof. We first notice that if $\sup(A \cdot B) = \infty$, then either $A$ or $B$ is unbounded; hence either $\sup(A) = \infty$ or $\sup(B) = \infty$, and as the other cannot be 0 for $\sup(A \cdot B)$ to be positive, the inequality holds. Likewise if $\sup(A) = \infty$ or $\sup(B) = \infty$ the inequality is immediate. Therefore, let us assume that $A \cdot B = \{ab : a \in A, b \in B\}$ is bounded above.

By the $\epsilon$ property of suprema, for all $\epsilon > 0$ there is an $ab \in A \cdot B$ such that $\sup(A \cdot B) \leq ab + \epsilon$

Note that the choice of $ab$ depends on $\epsilon$! Because $A, B \subseteq \mathbb{R}^+$, we are free to bound $a$ and $b$ above by their respective suprema without worrying about sign changes, so

$$\leq a \cdot \sup(B) + \epsilon$$

$$\leq \sup(A) \cdot \sup(B) + \epsilon$$

This last inequality holds for all $\epsilon > 0$. Because we have eliminated $ab$ from the equation, which depended on $\epsilon$, we are free to use the upper bound approximation technique to conclude,

$$\sup(A \cdot B) \leq \sup(A) \cdot \sup(B)$$

as desired. \qed

This technique will appear again and again in analysis through metric spaces, measure theory, and beyond.

Note that the assumption that $A \leq B + \epsilon$ in the technique hold for all $\epsilon > 0$ was not strictly necessary. A weaker condition with the same conclusion is that the set $E$ where $\epsilon \in E$ implies $A \leq B + \epsilon$ have an accumulation point at 0; that is, there are arbitrarily small elements of $E$. In practice, this means that we will either consider all $\epsilon > 0$ or $\epsilon = 1/n$, $n \in \mathbb{N}$. 