INVESTIGATION OF THE KÖBE-BIEBERBACH THEOREM

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1. Introduction

Complex analysis of functions uses the most natural of definitions to derive elegant results. An understanding of complex analysis sheds light into many other areas of analysis, such as topology and calculus. Many of the results of complex analysis are often simple, but with far-reaching implications, and all of these are from the most basic of assumptions.

Among the many mathematicians to study complex analysis, we are particularly interested in a result of Paul Köbe (1882-1945). Köbe, most known for his uniformization of the Riemann surface, also investigated what is now called the Köbe class of functions.

Definition 1.1. The **Köbe functions** are holomorphic, injective functions, for $0 \leq \theta < 2\pi$, of the form

$$f_\theta(z) \equiv \frac{z}{(1 + e^{i\theta} z)^2},$$

such that $f(0) = 0$, $f'(0) = 1$, and $|a_n| = n$ for all n in the power series $f(z) = a_n x^n + a_{n-1} x^{x-1} + \cdots + a_1 x + a_0$ [3].

The Köbe functions are a specific case of a more general class of functions, the *schlicht* functions, which will be considered throughout this paper. Ludwig Bieberbach (1886-1982), building off the work of Köbe [1], conjectured that all *schlicht* functions are such that $|a_n| \leq n$. Additionally, Köbe claimed that Köbe functions are the only *schlicht* functions such that $|a_n| = n$. Both Bieberbach's conjecture and Köbe's claim were proved by Louis de Branges in 1985 [2]. The work of de Branges is not necessary for proving the Köbe 1/4 Theorem (one of the oldest results in the history of *schlicht* functions), as Köbe had proved this many years prior to de Branges. This paper does not completely

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follow de Branges' proof, which drew heavily from operator theory, in proving the Köbe 1/4 Theorem.

Theorem 1.2 (Köbe 1/4 Theorem,1916)**.** *If f is schlicht, then* $D_{1/4}(0) \subset f(D_1(0))$ [3].

In this paper we will first recall some basic properties of complex functions, defining notation and, citing without proof, two standard theorems of complex analysis. We will then more closely examine the necessary parts of the hypothesis of the Köbe 1/4 Theorem in Section 2. In Section 3 we prove the Köbe 1/4 Theorem, following Stein and Shakarachi's guided outline of the proof [5]. This guided proof will include the results of Köbe's claim and Bieberbach's Conjecture. Finally, in Section 4, we note the implications of this theorem.

2. Background

We will be considering open sets in the complex plane $\mathbb{C}$. We denote the open disc of radius r, centered at $z_0 \in \mathbb{C}$ by $D_r(z_0) = \{z \mid |z - z_0| < r\}$, and denote the unit disc by $\mathbb{D} = D_1(0) = \{z \mid |z| < 1\}$.

In $\mathbb{R}^2$, it is of interest when a function is differentiable. Similarly, in the complex plane $\mathbb{C}$, we are interested in holomorphic functions, sometimes requiring the function to be injective.

Definition 2.1. A function f is **holomorphic** if the quotient

$$\frac{f(z_0 + h) - f(z_0)}{h}$$

converges to a limit as $h \to 0$, for all $z_0 \in \mathbb{C}$ [5].

Definition 2.2. A function on an open set is **univalent** if it is analytic (i.e. holomorphic) and injective [2].

The conjectures of Köbe and Bieberbach have hypotheses which require the functions under consideration to be *schlicht*.

Definition 2.3. A function of the unit disc $\mathbb{D}$ is called **schlicht** if

* f is univalent,
* $f(0) = 0$,
* $f'(0) = 1$.

Such a function has a power series of the form $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ [3].

The Köbe 1/4 Theorem considers the image of discs under a specific map, f, making use of a simple yet very useful theorem in complex analysis − the Open Mapping Theorem.

Theorem 2.4 (Open Mapping Theorem). *If f is holomorphic and nonconstant in a region Ω, then f is open* [5].

The Open Mapping Theorem tells us that, given a small open neighborhood around a point z_0, there is a small open neighborhood around the image $f(z_0)$ of that point under our map. We also will use the following theorem.

Theorem 2.5 (Jordan Curve Theorem). *Any continuous simple closed curve in the plane separates the plane into two disjoint regions, a bounded portion and an unbounded portion.*

3. Examination of the Hypotheses

We can restate the Köbe 1/4 Theorem in the following way using the terminology of the guided proof in Stein and Shakarchi:

Theorem 3.1. *Consider an injective holomorphic map on the unit disc $f : \mathbb{D} \to \mathbb{C}$ which satisfies $f(0) = 0$ and $f'(0) = 1$. Then there exists an $r > 0$ such that for all $f : \mathbb{D} \to \mathbb{C}, \quad D_r(0) \subset f(\mathbb{D})$, where the best possible value for r is $1/4$* [5].

The theorem requires that a function be *schlicht*, therefore satisfying the three conditions in Definition 2.3. But is this necessary? Let us individually consider the requirements of *schlicht* functions, also examining the Köbe class of *schlicht* functions.

Consider the case when $f(z)$ is only required to satisfy that $f(0) = 0$. By the Open Mapping Theorem, the image $f(\mathbb{D})$ must contain a small disc centered at the origin. We may ask then if there exists an $r > 0$ such that $D_r(0) \subset f(\mathbb{D})$, for all such f. We examine why no such r exists.

Consider the sequence of functions $\{f_n\}$, where

$$f_n(z) = \frac{z}{n},$$

for $n = \{1, 2, 3, \ldots\}$ and $z \in \mathbb{D}$. The function $f(z)$ is holomorphic, since z is differentiable everywhere in $\mathbb{C}$. Also, for all n,

$$f_n(0) = \frac{0}{n} = 0.$$

Since $f(z)$ is holomorphic, and therefore continuous, for each $r > 0$, there exists an $N \in \mathbb{Z}$ such that $f_N(\mathbb{D}) \subset D_r(0)$. Hence, if the hypothesis only required $f(0) = 0$ and $f(z)$ is holomorphic, then it is not assured that there is an open disc around zero.

Note in this last example $f_n'(0) = 1/n$, so as $n \to \infty$, $f_n'(z) = 0$. A necessary condition of the hypothesis is that $f'(0) = 1$. If we now require $f'(0) = 1$, we see the hypotheses are still not sufficient for proving the Köbe 1/4 Theorem.

Example 3.2. Assume that $f(z)$ is holomorphic, $f(0) = 0$ and $f'(0) = 1$. Consider the sequence of functions

$$f_\epsilon(z) = \epsilon(e^{z/\epsilon} - 1) = \epsilon e^{z/\epsilon} - \epsilon,$$

for $\epsilon > 0$. Since $\epsilon e^{z/\epsilon}$ will never equal zero, $f_\epsilon(z)$ will never equal $-\epsilon$. Since $f_\epsilon(0) = 0$, by the Open Mapping Theorem, $f(z)$ takes an open neighborhood around zero in $\mathbb{D}$ to an open neighborhood around zero in $\mathbb{C}$. However, the image never contains $-\epsilon$. Therefore, for any $r > 0$, there exists an $\epsilon > 0$ such that $f_\epsilon(\mathbb{D}) \subset D_r(0)$.

The hypothesis is not yet sufficient to prove the desired theorem. In the next section we assume that $f(z)$, a holomorphic function, be *schlicht*, which will allow us to prove the Köbe 1/4 Theorem.

4. Proof of the Köbe-Bieberbach Theorem

Conjecture 4.1. *If* $h(z) = 1/z + c_0 + c_1 z + c_2 z^2 + \cdots$ *is univalent for* $0 < |z| < 1$, *then*

$$\sum_{n=1}^{\infty} n|c_n|^2 \leq 1.$$

Proof. If $h(z) = 1/z + c_0 + c_1 z + c_2 z^2 + \cdots$ is analytic and injective for the given z, then $\lim_{z\to 0}|h(z)| \to \infty$. Hence, $h(z)$ has a pole at zero. We will then just consider $h(D_\rho(0) - \{0\})$, where $0 < \rho < 1$.

Let γ_ρ be a curve parametrized by $\theta \to h(\rho e^{i\theta})$ for $0 \leq \theta \leq 2\pi$. Since $h(z)$ is injective, h takes γ to a simple closed curve in $\mathbb{C}$. We parametrize γ by $\theta \to h(\rho e^{-i\theta})$. By the Jordan Curve Theorem, γ_ρ separates $\mathbb{C}$ into two components, one bounded and one unbounded. Since h has a pole at zero, then $h(D_\rho(0) - \{0\})$ is the unbounded component, for $\{z \mid 0 < |z| < 1\}$. We would like to compute the area of the bounded component, which we'll denote G_ρ. Note that the boundary of G_ρ is γ_ρ.

Using a derivation of the Gauss-Greene Theorem [4], the area can be found by

$$\text{Area}(G_\rho) = \int_{G_\rho} du \wedge dv = \int_{\partial G_\rho} udv = \int_{\gamma_\rho} udv.$$

Using complex notation, we let $w = u + iv$. Then $d\overline{w} \wedge dw = 2idu \wedge dv$, giving

$$\text{Area}(G_\rho) = \frac{1}{2i}\int_{G_\rho} d\overline{w} \wedge dw = \frac{1}{2i}\int_{\gamma_\rho} \overline{w}dw.$$

$$\frac{-1}{2i}\int_0^{2\pi}\overline{w}dw.$$

For $w(z) = h(\rho e^{i\theta})$, we have

$$\frac{-1}{2i}\int_0^{2\pi}\overline{h(\rho e^{i\theta})}h'(\rho e^{i\theta})i\rho e^{i\theta}d\theta.$$

Using $h(z)$ as defined above,

$$\frac{-1}{2i}\int_0^{2\pi}\Bigg[\left(\frac{1}{\rho e^{-i\theta}} + c_0 + c_1(\rho e^{-i\theta}) + c_2(\rho e^{-i\theta})^2 + \cdots\right)$$
$$\left(\frac{-1}{(\rho e^{i\theta})^2} + c_1 + 2c_2(\rho e^{i\theta}) + 3(\rho e^{i\theta})^2 + \cdots\right) i\rho e^{i\theta}d\theta$$

$$\Rightarrow \frac{-1}{2i}\int_0^{2\pi}\Bigg[\left(\frac{1}{\rho e^{-i\theta}} + c_0 + c_1(\rho e^{-i\theta}) + c_2(\rho e^{-i\theta})^2 + \cdots\right)$$
$$i\left(\frac{-1}{(\rho e^{i\theta})} + c_1(\rho e^{i\theta}) + 2c_2(\rho e^{i\theta})^2 + 3(\rho e^{i\theta})^3 + \cdots\right) d\theta.$$

To move the integral inside the summation, our function must be uniformly convergent. A product of holomorphic functions is a holomorphic function, and since holomorphic functions are uniformly convergent, we can move the integrand inside the summation. This is an area, therefore the integral is greater than zero. Noting that $|e^{i\theta}| = 1$, we reduce this to the following:

$$\frac{1}{\rho^2} - \sum_{k=1}^{\infty} k|a_k|^2\rho^{2k} \geq 0.$$

As $\rho \to 1$, we get that

$$1 \geq \sum_{k=1}^{\infty} k|a_k|^2.$$

□

Thus we see that the area of the complement of $h(D_\rho(0) - \{0\})$ is bounded. Through the next three conjectures, we will relate this back to our original function $f(z)$ to get the desired result of bounding the image of a disc around zero with a lower bound. We will next look at a reformulation of Robertson's Conjecture [2].

Conjecture 4.2. *If $f(z) = z + a_2z^2 + \cdots$ is a holomorphic, schlicht function, then there exists a schlicht function $g(z)$ such that*

$$[g(z)]^2 = f(z^2).$$

Proof. We want a function $\psi(z)$, such that ψ is holomorphic, and $\psi(0) = 1$. If such a ψ exists, then $\psi^2(z)$ exists. (We will show in a moment that such a $\psi(z)$ does in fact exist.)

Note that $f(z)/z$ is nowhere vanishing. Let $\frac{f(z)}{z} = \psi^2(z)$, where $\psi(z)$ is holomorphic. So long as $\psi(0) = 1$, meaning $\psi(z)$ is kept away from zero, then $\psi^2(z)$ is holomorphic. Let $g(z) = z\psi^2(z)$. Then

$$\psi^2(z) = f(z)/z \Rightarrow \psi(z) = \sqrt{\frac{f(z)}{z}}.$$

At z^2,

$$\psi(z^2) = \sqrt{\frac{f(z^2)}{z^2}} \Rightarrow \psi(z^2) = \frac{1}{z}\sqrt{f(z^2)}$$
$$\Rightarrow z\psi(z^2) = \sqrt{f(z^2)} \Rightarrow z^2\psi^2(z^2) = f(z^2) \Rightarrow [g(z)]^2 = f(z^2).$$

Therefore we see that such a ψ exists. However, is $g(z)$ a *schlicht* function? Assume $g(z) = g(w)$. Then

$$[g(z)]^2 = [g(w)]^2 \Rightarrow f(z^2) = f(w^2) \Rightarrow z^2 = w^2 \Rightarrow z = \pm w.$$

Note that g is an even function, meaning that all of the even terms in the expansion of g are zero. Therefore if $z = -w$, then for $g(z) = g(w) = g(-z)$, z and w would have to equal zero. Therefore $z = w$.

Since $\psi(z)$ is holomorphic, we can write it as a power series:

$$\frac{f(z)}{z} = \psi(z)\psi(z) = 1 + a_2 z + a_3 z^2 + \cdots,$$
$$\Rightarrow$$
$$\sum_{i=0}^{\infty} b_i z^i \sum_{i=0}^{\infty} b_i z^i = 1 + a_2 z + a_3 z^2 + \cdots.$$

Since $g(z)$ is injective, $b_0 = 1$, $b_1 = a_2/2$, etc. Additionally, $g(0) = 0$ and $g'(0) = \psi^2(0) + z(\psi^2)'(0) = 0 + (1 + 0) = 1$, so we see that $g(z)$ is *schlicht*. □

In the proof of Conjecture 4.2, we took the square root of $f(z)/z$. This is only possible by the following theorem.

Theorem 4.3. *Suppose Ω is simply connected; f is holomorphic on Ω and never vanishes. Then there exists a holomorphic function $w(z)$ such that $f(z) = e^{w(z)}$.*

The function f is holomorphic and never vanishing on our simply connected set, so such a $w(z)$ exists. We can then define $sqrt f(z) = e^{\frac{1}{2}w(z)}$. Letting $w(z) = h(\rho e^{i\theta})$, allowed us to finish the proof of the theorem.

We now prove another preliminary conjecture to the Köbe 1/4 Theorem.

Conjecture 4.4. *If* $f(z) = z + a_2 z^2 + \cdots$ *is a holomorphic, schlicht function, and* $g(z)$ *is a schlicht function such that* $[g(z)]^2 = f(z^2)$, *then* $|a_2| \leq 2$, *and*

$$|a_2| = 2, \iff f(z) = \frac{z}{(1 - e^{i\theta} z)^2}.$$

Bieberbach's Conjecture is actually more general.

Theorem 4.5 (Bieberbach Conjecture, 1916)**.** *If* f *is a schlicht function and has the power series representation*

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots ,$$

then $|a_n| \leq n$. *If there is some integer* n *such that* $|a_n| = n$, *then* f *is a rotation of the Köbe function* [2].

Proof. (Conjecture 4.3) Since $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$, then we know $[g(z)]^2$:

$$f(z^2) = z^2 + a_2 z^4 + a_3 z^6 + \ldots = g^2(z).$$

Note that $[g(z)]^2$ has no constant terms , therefore $g(z)$ has no constant terms, hence $1/g(z)$ has no constant terms.

We already know that $g(z)$ is injective. Therefore $1/g$ has a pole at zero, so can be written as

$$\frac{1}{g} = \frac{1}{z} + G(z),$$

where $G(z)$ is holomorphic. Then we have

$$\frac{1}{[g(z)]^2} = \frac{1}{z^2} + \frac{2}{z} G(z) + [G(z)]^2 = \frac{1}{z^2} + G'(z),$$

where $G'(z) = \frac{2}{z} G(z) + [G(z)]^2$. Since $G(z)$ has no constant terms, then $\frac{2}{z} G(z)$ is still holomorphic.

To determine $|a_2|$, we calculate the coefficients.

$$\frac{1}{[g(z)]^2} = \frac{1}{z^2} + G'(z) = \frac{1}{z^2} + \sum_{k=0}^{\infty} b_k z^k.$$

Note that $\frac{1}{[g(z)]^2} * [g(z)]^2 = 1$, and we know the terms of $[g(z)]^2$, so we can write

$$(\frac{1}{z^2}+\sum_{k=0}^{\infty} b_k z^k)(z^2+a_2z^4+a_3z^6+\cdots)=1.$$

We can thus solve for the b_k's, knowing that the constant term of the product must equal one and all other coefficients must equal zero: (a_2+b_0) must equal zero, therefore $b_0=-a_2$, and $b_1=0$, and $b_2=a_2^2-a_3$. The rest of the coefficients can be calculated in a like manner.

Therefore $G'(z)=\frac{2}{z}G(z)+G^2(z)=b_0+b_2z^2+\cdots$. We multiply through by z, and get

$$2G+zG^2=b_0z+b_2z^3+\cdots\Rightarrow G(2+zG)=\sum_{k=0}^{\infty}c_kz^k(2+\sum_{k=0}^{\infty}c_kz^{k+1})=b_0z+b_2z^3+\cdots.$$

Therefore $c_0=0$, $2c_1=\frac{b_0}{2}=\frac{-a_2}{2}$, etc.

By Conjecture 4.1, we know that $\sum_{n=1}^{\infty}n|c_n|^2\leq 1$, therefore

$$|c_1|^2\leq 1\Rightarrow(|a_2|^2)/4\leq 1\Rightarrow|a_2|^2\leq 4\Rightarrow|a_2|\leq 2.$$

Since the sum of the terms is less than or equal to one, and $|c_1|^2\leq 1$, then all other terms are equal to zero.

Finally, note that

$$|a_2|=2\Rightarrow a_2=|a_2|e^{i\theta}=2e^{i\theta}$$

$\Leftrightarrow\frac{1}{g}=\frac{(1-e^{i\theta}z^2)}{z}\Leftrightarrow\frac{1}{g^2}=\frac{(1-e^{i\theta}z^2)}{z}^2\Leftrightarrow\frac{1}{g^2}=\frac{(1-e^{i\theta}z^2)^2}{z^2}=\frac{1}{f(z^2)}\Leftrightarrow\frac{1}{f}=\frac{(1-e^{i\theta}z)^2}{z}\Leftrightarrow f=\frac{z}{(1-e^{i\theta}z)^2}$

□

Now that we have the Bieberbach conjecture, we quickly move towards proving the Köbe 1/4 Theorem.

Conjecture 4.6. *If $h(z)=\frac{1}{z}+c_0+c_1z+c_2z^2+\cdots$ is injective on $\mathbb{D}$ and avoids the values z_1 and z_2, then $|z_1-z_2|\leq 4$.*

Proof. Consider the function

$$\phi(z)=\frac{1}{h(z)-z_1}.$$

Since $h(z)$ is injective and avoids z_1, then $\phi(z)$ is injective. Since $h(z)$ has a removable singularity at zero, with $\phi(0)=0$, and $\phi'(0)=1$, then ϕ is *schlicht*. By Conjecture 4.5, the power series expansion of $\phi(z)$, a holomorphic function, has $|\alpha_2|\leq 2$, when $\phi(z)=\sum_{i=0}^{\infty}\alpha_iz^i$. Thus

$$\Rightarrow \frac{1}{h(z)-z_i} = \sum_{i=0}^{\infty} \alpha_i z^i$$
$$\Rightarrow 1 = (h(z)-z_1)(\sum_{i=0}^{\infty} \alpha_i z^i)$$
$$\Rightarrow 1 = (\frac{1}{z} + c_0 + c_1 z + c_2 z^2 + \ldots + z_1)(\sum_{i=0}^{\infty} \alpha_i z^i)$$
$$\Rightarrow 1 = (\frac{1}{z} + (c_0 + z_1) + c_1 z + c_2 z^2 + \ldots)(\sum_{i=0}^{\infty} \alpha_i z^i).$$

Calculating the coefficients, we have that $\alpha_0 = 0$, since there are no $1/z$ terms, $\alpha_1 = 1$, and $\alpha_2 = -z_1 - c_0$. Then

$$|\alpha_2| \leq 2 \Rightarrow |z_1 - c_0| \leq 2.$$

Similarly, for z_2,

$$|\alpha_2| \leq 2 \Rightarrow |z_2 - c_0| \leq 2.$$

Therefore

$$|z_1 - z_2| = |(z_1 - c_0) + (c_0 - z_2)| \leq |z_1 - c_0| + |c_0 - z_2| \leq 2 + 2 = 4.$$

□

Conjecture 4.7. *Given the above hypotheses,* $D_{1/4}(0) \subset f(\mathbb{D})$.

Proof. Assume $f(z)$, holomorphic, avoids some point $w \in \mathbb{C}$. Then $1/f(z)$ avoids $1/w$ and 0. By Conjecture 4.6, $1/f(z) \leq 4 \Rightarrow f(z) \leq 1/4$. Then f acting on $z \in \mathbb{D} \Rightarrow f(\mathbb{D}) \subset D_{1/4}(0)$.

□

Therefore if f avoids some point $w \in \mathbb{C}$, then w is outside of $D_{1/4}(0)$.

5. Conclusion

We have thus proved the Köbe 1/4 Theorem. This collective Köbe-Bieberbach Theorem tells us that for any *schlicht* function, the image of the function must contain a disc of minimal size. This will be useful in our next section on mapping problems.

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