Math 410 HW 1 Solution:s

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1. Consider the function f defined on the unit interval $[0,1]$ by

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ x\sin(1/x) & \text{if } 0 < x \leq 1. \end{cases}$$

(a) Is f continuous on $[0,1]$?
Solution: The function f as defined on the half-open interval $(0,1]$ is obviously continuous, being the composition and product of continuous well-defined functions. We therefore only need to check continuity of f at $x = 0$.
Note that $-x \leq x\sin(1/x) \leq x$ so that the squeeze theorem implies

$$\lim_{x\to 0^+} f(x) = 0 = f(0).$$

This guarantees that f is also continuous at $x = 0$. Thus, f is indeed continuous on $[0,1]$.

(b) Does f belong to $\mathcal{C}^1[0,1]$?
Solution: Standard calculus reveals that for x in the ***open*** interval $(0,1)$,

$$f'(x) = -\frac{1}{x}\cos\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x}\right),$$

which is continuous. However, there is ***no possible way*** to guess the value of $f'(0)$ without doing some work. Using the definition of the derivative,

$$\begin{aligned} f'(0) &= \lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h\to 0^+} \frac{f(h) - 0}{h} \\ &= \lim_{h\to 0^+} \frac{h\sin(1/h)}{h} \\ &= \lim_{h\to 0^+} \sin(1/h), \end{aligned}$$

which does not exist. Hence, f does not have a derivative (much less a continuous derivative) at $x = 0$, so f does not belong to $\mathcal{C}^1[0,1]$.

2. Find a function $g \in C^\infty(\mathbb{R})$ which satisfies

$$g(x) = \begin{cases} 0 & \text{if } x \leq -1 \\ 1 & \text{if } x \geq 0. \end{cases}$$

Hint: Suppose $\phi \in C^\infty(\mathbb{R})$ has supp $\phi \subset [-1, 0]$ and $\phi(x) \geq 0$ for all x. Consider the properties of $\psi(x) = \int_{-\infty}^{x} \phi(y)dy$.

Solution: Recall the example of a function in $C^\infty(\mathbb{R})$ given in class:

$$h(x) = \begin{cases} 0 & x \leq 0 \\ e^{-\frac{1}{x}} & x > 0. \end{cases}$$

Following the hint, we will modify $h(x)$ to meet the requirements of ϕ. Consider the polynomial $p(x) = -x^2 - x = -x(x+1)$. Observe that $p(0) = p(-1) = 0$, and $p(x) > 0$ for $x \in (-1, 0)$. Let

$$\phi(x) = h(p(x))$$

Since ϕ is the composition of two C^∞ functions, it will also be C^∞. For $x \geq 0$ or $x \leq -1$, $p(x) \leq 0$, so $h(p(x)) = 0$. Therefore, supp $\phi \subset [0, 1]$.

Now, let

$$\psi(x) = \int_{-\infty}^{x} \phi(y)dy.$$

By FTC, since $\phi \in C^\infty$, so is ψ.
For $x \leq -1$, $\psi(x) = 0$ since $\phi(y) = 0$ for all $y \leq -1$.
For $x \in (-1, 0)$, $\psi(x) = \int_{-1}^{x} \phi(y)dy > 0$, since $\phi(y) > 0$ for $y \in (-1, 0)$.
For $x \geq 0$, $\psi(x) = \int_{-1}^{0} \phi(y)dy$. To scale this to be equal to one, let $C = \int_{-1}^{0} \phi(y)dy$ and let $g(x) = \frac{\psi(x)}{C}$. Then g has all the desired properties.

3. Let $I = [a, b]$ where $a < b$. Suppose $\varepsilon > 0$. Find a function $\phi \in \mathcal{C}^\infty(\mathbb{R})$ such that $\phi(x) = 1$ if $x \in I$ and supp $\phi \subset (a - \varepsilon, b + \varepsilon)$.

Solution: Using the $g \in \mathcal{C}^\infty(\mathbb{R})$ we created from the previous problem, define ϕ by:

$$\phi(x) = \begin{cases} g(\frac{x-a}{\epsilon/2}) & x \leq \frac{a+b}{2} \\ g(\frac{b-x}{\epsilon/2}) & x \geq \frac{a+b}{2} \end{cases}$$

Then $\phi(a - \varepsilon/2) = g(-1) = 0$, $\phi(a) = g(0) = 1$, and $\phi(x) = 1$ for $a < x \leq (a+b)/2$. Similarly, $\phi(b) = g(0) = 1, \phi(b + \varepsilon/2) = g(-1) = 0$, and $\phi(x) = 1$ for $(a+b)/2 \leq x < b$. Also $\phi(x)$ will be equal to zero if $x \leq a - \varepsilon/2$ or $x \geq b + \varepsilon/2$.

Therefore, $\phi \in \mathcal{C}^\infty(\mathbb{R})$ such that $\phi(x) = 1$ if $x \in I$, and supp $\phi \subset (a - \epsilon, b + \epsilon)$.

4. Analyze the variational problems corresponding to the following functionals, where in each case, $y(0) = 0$ and $y(1) = 1$. Does a minimum or a maximum exist?

   (a) $\int_0^1 y'\,dx$
   Solution: Evaluating the integral,

   $$\int_0^1 y'\,dx = y(1) - y(0) = 1 - 0 = 1.$$

   The functional is therefore constant, so every admissible function y is an extremal function, by virtue of being both a maximum and a minimum. Saying that there is no extremum is technically incorrect.

   (b) $\int_0^1 yy'\,dx$
   Solution: Integrating by parts,

   $$\int_0^1 yy'\,dx = y^2|_0^1 - \int_0^1 yy'\,dx.$$

   Thus,

   $$\int_0^1 yy'\,dx = \frac{1}{2}\left(y^2(1) - y^2(0)\right) = \frac{1}{2}.$$

   The functional is therefore constant, so every admissible function y is a minimum and a maximum.

   (c) $\int_0^1 xyy'\,dx$
   Solution using ad hoc methods: Integrating by parts,

   $$\int_0^1 xyy'\,dx = \frac{1}{2}(xy^2|_0^1 - \int_0^1 y^2dx) = \frac{1}{2}(1 - \int_0^1 y^2dx).$$

   As $y^2(x) \geq 0$, it is clear that $\int_0^1 y^2dx \geq 0$. So the maximal value our functional can possibly achieve is $1/2$, but we do not know if any admissible function achieves that value.

   With that in mind, consider the function $y(x) = x^k$. Then

   $$\int_0^1 xyy'dx = \frac{1}{2}\left(1 - \int_0^1 x^{2k}dx\right) = \frac{1}{2}\left(1 - \frac{1}{2k+1}\right)$$

   As k gets larger, this value gets closer to $1/2$. To actually equal $1/2$, $k \to \infty$, but in terms of our function, this would mean $y(x) = 0$ for $x \in [0, 1)$ and $y(1) = 1$. However, this function is not continuous, and thus the functional **does not have a maximum** when restricted to continuous functions with designated boundary conditions.

   As far as a minimum goes, $\int_0^1 y^2dx$ can be as large as we like, making our functional value as small as we'd like, meaning **a minimum also does not exist**.

For a specific example, consider $y(x) = kx(x-1)$. Then $\int_0^1 y^2 dx = k^2/30$.

Solution using Euler-Lagrange Equation: The Euler-Lagrange equation associated to the given functional is

$$0 = xy' - \frac{d}{dx}(xy) = xy' - y - xy' = -y.$$

This would mean that any extremum would have to satisfy $y(x) = 0$ for all $x \in [0,1]$. However, none of our admissible functions satisfy this condition, so there are no extrema.

5. Find the extremals of the following functionals:

   (a) $\int_a^b (y^2 + y'^2 - 2y\sin(x))dx$
   Solution: $F(x, y, y') = y^2 + y'^2 - 2y\sin(x)$, so the Euler equation is

   $$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = (2y - 2\sin(x)) - (2y'') = 0.$$

   $$\implies y''(x) = y(x) - \sin(x)$$

   Solving this differential equation gives us the general solution

   $$y(x) = C_1 e^x + C_2 e^{-x} + \frac{\sin(x)}{2}$$

   for constants C_1, C_2.

   (b) $\int_a^b \frac{y'^2}{x^3} dx$
   Solution: The associated Euler equation is

   $$\frac{d}{dx}\left(\frac{2y'}{x^3}\right) = 0$$

   $$\implies \frac{2y'}{x^3} = C_1$$

   $$\implies 2y' = C_1 x^3$$

   $$\implies y = C_2 x^4 + C_3$$

   (c) $\int_a^b (y^2 - y'^2 - 2y\cosh(x))dx$
   Solution: The associated Euler equation is

   $$(2y - 2\cosh(x)) - (-2y'') = 0$$

   $$\implies y'' = -y + \cosh(x)$$

   The general solution to this differential equation is

   $$y(x) = \cosh(x)/2 + C_1\cos(x) + C_2\sin(x).$$