1 Homework #10

1. Consider the functional with Lagrangian

$$F(x,u,p) = \sqrt{x^2+u^2}\sqrt{1+p^2}.$$

(a) Find the Hamiltonian and the associated Hamiltonian system.

Solution Find that $\pi = \frac{\partial F}{\partial p} = \frac{p\sqrt{x^2+u^2}}{\sqrt{1+p^2}}$, so that (after simplifying), the Hamiltonian is

$$H(x,u,\pi) = \pi p - F(x,u,\pi) = -\sqrt{x^2+u^2-\pi^2}.$$

The Hamiltonian system is

$$\frac{du}{dx} = \frac{\partial H}{\partial \pi} = \frac{\pi}{\sqrt{x^2+u^2-\pi^2}},$$

$$\frac{d\pi}{dx} = -\frac{\partial H}{\partial u} = \frac{u}{\sqrt{x^2+u^2-\pi^2}}.$$

(b) Show that $u^2-\pi^2$ is a conserved quantity.

Solution Using the chain rule,

$$\frac{d}{dx}\left(u^2-\pi^2\right) = 2u\frac{du}{dx} - 2\pi\frac{d\pi}{dx}.$$

From part (a), this derivative is zero, so $u^2-\pi^2$ is indeed conserved.

(c) Find a solution to the Hamilton-Jacobi equation with the form

$$S(x,u) = \frac{1}{2}\left(Ax^2+2Bxu+Cu^2\right).$$

Solution Using $S_x = Ax+Bu$ and $S_u = Bx+Cu$, the Hamilton-Jacobi equation becomes

$$0 = S_x + H(x,u,S_u) = Ax+Bu-\sqrt{x^2+u^2-(Bx+Cu)^2}.$$

Solve this to find that $A=-C$ and $B=\sqrt{1-A^2}$. A general solution of the given form is then

$$S(x,u) = \frac{1}{2}\left(Ax^2+\sqrt{1-A^2}xu-Au^2\right).$$

(d) Show that the solutions are given implicitly in the form

$$(x^2-u^2)\cos\alpha + 2ux\sin\alpha = C,$$

where α and C are constants.

Solution The extremals will satisfy $S_A = \frac{1}{2}\left(x^2-u^2-\frac{2A}{\sqrt{1-A^2}}ux\right)$ constant. With $A=\sin\alpha$, find that $x^2-u^2-\frac{\sin\alpha}{\cos\alpha}$ is constant, so multiplying this by the constant $\cos\alpha$ finishes the problem.

2. Consider a particle with mass m in $\mathbb{R}^3$ moving under the influence of the potential V. Of course, the Lagrangian is

$$L(x, y, z, p, q, r) = \frac{m}{2}(p^2 + q^2 + r^2) - V(x, y, z).$$

(a) Write the Lagrangian in polar coordinates r, θ, and ϕ, defined by $x = r\sin\phi\cos\theta$, $y = r\sin\phi\sin\theta$, and $z = r\cos\phi$.

Solution By calculations similar to Homework 3 Exercise 3, find that

$$p^2 + q^2 + r^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \dot{r}^2 + r^2\dot{\phi}^2 + r^2\sin^2\phi\,\dot{\theta}^2.$$

The new Lagrangian is then

$$L(r, \theta, \phi, \dot{r}, \dot{\theta}, \dot{\phi}) = \frac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2 + r^2\sin^2\phi\,\dot{\theta}^2) - V(r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\phi).$$

(b) What is the corresponding Hamiltonian?

Solution Find that

$$\pi = \frac{\partial L}{\partial \dot{r}} = m\dot{r},\ \sigma = \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi},\ \text{and}\ \rho = \frac{\partial L}{\partial \dot{\theta}} = mr^2\sin^2\phi\,\dot{\theta}.$$

By definition, the Hamiltonian is

$$H(r, \theta, \phi, \pi, \rho, \sigma) = \pi\dot{r} + \sigma\dot{\phi} + \rho\dot{\theta} - L(r, \theta, \phi, \pi, \rho, \sigma),$$

which with the above, simplifies to

$$H(r, \theta, \phi, \pi, \rho, \sigma) = \frac{1}{2m}\left(\pi^2 + \frac{\sigma^2}{r^2} + \frac{\rho^2}{r^2\sin^2\phi}\right) + V(r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\phi).$$

(c) Assuming that $V = -\frac{k}{r}$, reduce the solution of the Hamilton-Jacobi equation to the evaluation of integrals.

Solution The new Hamilton-Jacobi equation is

$$S_t + H(r, \theta, \phi, S_r, S_\theta, S_\phi) = 0$$
$$S_t + \frac{1}{2m}\left(S_r^2 + \frac{S_\theta^2}{r^2\sin^2\phi} + \frac{S_\phi^2}{r^2}\right) - \frac{k}{r} = 0.$$

Assume a separable solution

$$S(t, r, \theta, \phi) = f(t) + g(r, \theta, \phi).$$

There must exist a constant α such that

$$f'(t) = -\alpha,\ \text{and}$$

$$\frac{1}{2m}\left(\left(\frac{\partial g}{\partial r}\right)^2 + \frac{1}{r^2\sin^2\phi}\left(\frac{\partial g}{\partial \theta}\right)^2 + \frac{1}{r^2}\left(\frac{\partial g}{\partial \phi}\right)^2\right) - \frac{k}{r} = \alpha. \tag{1}$$

It is clear that $f(t) = -\alpha t$.
Assume a separable solution

$$g(r,\theta,\phi) = h(r) + w(\theta,\phi).$$

Multiply Equation (1) by r^2 to find that

$$-\alpha r^2 + \frac{1}{2m}\left(r^2 h'(r)^2 + \frac{1}{\sin^2\phi}w_\theta^2 + w_\phi^2\right) - kr = 0.$$

With some more simplifying, there must exist a constant β such that

$$\frac{1}{2m}(h'(r)^2 - \alpha)r^2 - kr = -\beta, \text{ and} \tag{2}$$

$$\frac{1}{2m}\left(\frac{1}{\sin^2\phi}w_\theta^2 + w_\phi^2\right) = \beta. \tag{3}$$

Assume a separable solution

$$w(\theta,\phi) = u(\theta) + v(\phi).$$

Then from Equation (3),

$$\frac{1}{2m}(u'(\theta)^2 + \sin^2\phi\, v'(\phi)^2) - \beta\sin^2\phi = 0.$$

There exists a constant γ such that

$$\frac{1}{2m}u'(\theta)^2 = -\gamma \text{ and} \tag{4}$$

$$\sin^2\phi\left(\frac{1}{2m}v'(\phi)^2 - \beta\right) = \gamma. \tag{5}$$

Now to gather up the integrals.
From Equation (2),

$$h(r) = \int\sqrt{\frac{2m}{r^2}(kr-\beta) + \alpha}\,dr.$$

From Equation (4),

$$u(\theta) = \sqrt{-2m\gamma}\theta.$$

From Equation (5),

$$v(\phi) = \int\sqrt{2m\left(\frac{\gamma}{\sin^2\phi} + \beta\right)}\,d\phi.$$

With these three integrals, the solution to the Hamilton-Jacobi equation is $S(t,r,\theta,\phi) = -\alpha t + h(r) + u(\theta) + v(\phi)$.

3. Calculate the second variation of the functional

$$\mathcal{F} = \int_a^b F(x, u(x), u'(x))\, dx$$

under the following assumptions.

(a) $F(x, u, p)$ does not depend on u.
Solution By definition,

$$\delta^2 \mathcal{F}(u, \phi) = \int_a^b \left(F_{uu}\phi^2 + 2F_{up}\phi\phi' + F_{pp}\phi'^2 \right)\, dx.$$

With no u dependence,

$$\delta^2 \mathcal{F}(u, \phi) = \int_a^b F_{pp}\phi'^2\, dx.$$

(b) $F(x, u, p)$ does not depend on p.
Solution With no p dependence,

$$\delta^2 \mathcal{F}(u, \phi) = \int_a^b F_{uu}\phi^2\, dx.$$

4. Suppose $F(x, u, p) = A(x)u + B(x)p$, where A and B are functions depending only on x. Compute the first and second variations.

Solution The first variation is

$$\delta \mathcal{F}(u, \phi) = \int_a^b (F_u\phi + F_p\phi')\, dx.$$

It is clear that

$$\delta \mathcal{F}(u, \phi) = \int_a^b (A(x)\phi + B(x)\phi')\, dx,$$

and

$$\delta^2 \mathcal{F}(u, \phi) = 0.$$

5. Suppose that $F(x, u, p) = A(x) + B(x)u^2 + 2C(x)up + D(x)p^2$. Compute the first and second variations.

Solution Find that

$$\begin{aligned} F_u &= 2B(x)u + 2C(x)p \\ F_p &= 2C(x)u + 2D(x)p \\ F_{uu} &= 2B(x) \\ F_{up} &= 2C(x) \\ F_{pp} &= 2D(x). \end{aligned}$$

Then

$$\delta\mathcal{F}(u,\phi) = \int_a^b (2B(x)u + 2C(x)p)\phi) + 2C(x)u + 2D(x)p)\phi')\,dx,$$

and

$$\delta^2\mathcal{F}(u,\phi) = 2\int_a^b (B(x)\phi^2 + 2C(x)\phi\phi' + D(x)\phi'^2)\,dx.$$