Math 410 Homework Solutions

February 4, 2009

1 Homework #2

Find the extremal for the variational integral

$$\mathcal{F}(u) = \int_a^b F(x, u(x), u'(x))dx$$

subject to the boundary conditions $u(a) = c, u(b) = d$.

1. $F(x, u, p) = 2u + p^2, (a, c) = (0, 0)$ and $(b, d) = (1, 1)$.

$$F_u - \frac{d}{dx}F_p = 0$$

$$\Longrightarrow (2) - \frac{d}{dx}(2p) = 0$$

$$\Longrightarrow 2 - 2u'' = 0$$

$$\Longrightarrow u'' = 1$$

   The general solution to this differential equation is

$$u(x) = \frac{1}{2}x^2 + C_1 x + C_2$$

   Using boundary conditions, $u(0) = 0 \Longrightarrow C_2 = 0$, and $u(1) = 1, \Longrightarrow C_1 + 1/2 = 1$. Therefore,

$$u(x) = \frac{1}{2}(x^2 + x)$$

   .

2. $F(x, u, p) = p^2 + 2up, (a, c) = (-1, 1)$ and $(b, d) = (2, 0)$.

$$F_u - \frac{d}{dx}F_p = 0$$

$$\Longrightarrow 2p - \frac{d}{dx}(2p + 2u) = 0$$

$$\Longrightarrow 2u' - (2u'' + 2u') = 0$$
$$\Longrightarrow u'' = 0$$

The general solution is

$$u(x) = C_1 x + C_2$$

Using boundary conditions, $u(-1) = 1 \Longrightarrow C_2 - C_1 = 1, u(2) = 0 \Longrightarrow 2C_1 + C_2 = 0$. Together, this implies

$$u(x) = \frac{1}{3}(2 - x.)$$

3. $F(x, u, p) = p^2 + 2xp + x^2, (a, c) = (0, 0)$ and $(b, d) = (1, 0)$.

$$F_u - \frac{d}{dx}F_p = 0$$

$$\Longrightarrow -\frac{d}{dx}(2p + 2x) = 0$$
$$\Longrightarrow -(2u'' + 2) = 0$$
$$\Longrightarrow u'' = -1$$

The general solution is

$$u(x) = -1/2x^2 + C_1 x + C_2.$$

$u(0) = 0 \Longrightarrow C_2 = 0, u(1) = 0 \Longrightarrow C_1 = 1/2$.

$$u(x) = \frac{1}{2}(-x^2 + x)$$

.

4. $F(x, u, p) = u^2 + 2up + p^2, (a, c) = (0, 0)$ and $(b, d) = (2, 1)$.

$$F_u - \frac{d}{dx}F_p = 0$$

$$\Longrightarrow (2u + 2p) - \frac{d}{dx}(2u + 2p) = 0$$
$$\Longrightarrow (2u + 2u') - (2u' + 2u'') = 0$$
$$\Longrightarrow u'' = u$$

The general solution is

$$u(x) = C_1 e^x + C_2 e^{-x}$$

(or equivalently $u(x) = C_3 \cosh(x) + C_4 \sinh(x)$).

Using boundary conditions, $u(0) = 0 \implies C_1 = -C_2$ and $u(2) = 1 \implies C_1 = \frac{1}{e^2 - e^{-2}} = \frac{1}{2\sinh(2)}$.

$$u(x) = \frac{1}{2\sinh(2)}(e^x - e^{-x}) = \frac{\sinh(x)}{\sinh(2)}$$

5. $\int_0^{\pi/2} (y'^2 + z'^2 + 2yz)dx$

   A necessary condition for $(y(x), z(x))$ to be an extemal of the functional is that both y and z satisfy the Euler equations, i.e.

$$F_y - \frac{d}{dx} F_{y'} = 0$$
$$F_z - \frac{d}{dx} F_{z'} = 0$$

   For this functional, that means

$$2z - \frac{d}{dx}(2y') = 0$$
$$2y - \frac{d}{dx}(2z') = 0$$

$$\implies$$

$$2z - 2y'' = 0$$
$$2y - 2z'' = 0$$

$$\implies$$

$$y'' = z$$
$$z'' = y$$

   You can solve this system, or notice that this implies $y^{(4)} = y$. The general solution for this is

$$z(x) = y(x) = C_1 e^x + C_2 e^{-x} + C_3 \cos(x) + C_4 \sin(x)$$

   After using all four boundary conditions (and the fact that $z(x) = y''(x)$), the solution is

$$y(x) = \frac{1}{\sinh(\pi/2)} \left(\frac{e^x - e^{-x}}{2} \right) = \frac{\sinh(x)}{\sinh(\pi/2)}$$

6. $\int_{x_0}^{x_1} (y'^2 + z'^2 + y'z')dx$

$$F_y - \frac{d}{dx} F_{y'} = 0$$
$$F_z - \frac{d}{dx} F_{z'} = 0$$

For this functional, that means

$$-\frac{d}{dx}(2y' + z') = 0$$
$$-\frac{d}{dx}(2z' + y') = 0$$

$$\Longrightarrow$$

$$2y' + z' = C_1$$
$$2z' + y' = C_2$$

$$\Longrightarrow$$

$$y' = C_3 \Longrightarrow y(x) = C_3 x + C_4$$
$$z' = C_4 \Longrightarrow z(x) = C_5 x + C_6$$

7. $\int_{x_0}^{x_1} (2yz - 2y^2 + y'^2 - z'^2)dx$

$$F_y - \frac{d}{dx} F_{y'} = 0$$

$$F_z - \frac{d}{dx} F_{z'} = 0$$

For this functional, that means

$$(2z - 4y) - \frac{d}{dx}(2y') = 0$$
$$(2y) - \frac{d}{dx}(-2z') = 0$$

$$\Longrightarrow$$

$$2z - 4y - 2y'' = 0$$
$$2y + 2z'' = 0$$

$$\Longrightarrow$$

$$y = -z'', y'' = -z^{(4)}$$
$$z + 2z'' + z^{(4)} = 0$$

The general solution to the last equation is

$$z(x) = (C_1 + C_2x)\sin(x) + (C_3 + C_4x)\cos(x)$$

and from this, we get

$$y(x) = (C_1 + 2C_4 + C_2x)\sin(x) + (C_3 - 2C_2 + C_4x)\cos(x)$$

8. $\int_{x_0}^{x_1} F(y', z')dx$ given that $F_{y'y'}F_{z'z'} - F_{y'z'}^2 \neq 0$ for $x \in [x_0, x_1]$

$$F_y - \frac{d}{dx}F_{y'} = 0$$

$$F_z - \frac{d}{dx}F_{z'} = 0$$

For this functional, that means

$$\frac{d}{dx}F_{y'} = 0 \Longrightarrow F_{y'y'}y'' + F_{y'z'}z'' = 0$$
$$\frac{d}{dx}F_{z'} = 0 \Longrightarrow F_{z'z'}z'' + F_{z'y'}y'' = 0$$

From the first equation, $y'' = \frac{-F_{y'z'}}{F_{y'y'}}z''$ and substituting this into the second equation yields

$$\left(F_{z'z'}F_{y'y'} - F_{y'z'}^2\right)z'' = 0.$$

Since $F_{y'y'}F_{z'z'} - F_{y'z'}^2 \neq 0$, $z'' = 0$. Similarly, $y'' = 0$. Therefore, $y(x) = C_1x + C_2, z(x) = C_3x + C_4$.