1 Homework #3

1. Suppose $F(x,u,p) = u^2p^2$, and $\mathcal{F}(u) = \int_a^b F(x,u(x),u'(x))dx$

   (a) Show all extremals are parabolas.

   Solution The Euler equation for this functional is

$$2up^2 - \frac{d}{dx}(2u^2p) = 0$$

$$2up^2 - (4up^2 + 2u^2p') = 0$$

$$u(u')^2 + u^2u'' = 0$$

$$u\left((u')^2 + uu''\right) = 0$$

$$\Longrightarrow u = 0 \text{ or } (u')^2 + uu'' = 0$$

   Notice that $\left(u^2\right)' = 2uu'$ and $\left(u^2\right)'' = 2(u')^2 + 2uu''$. Therefore, either $u = 0$ or

$$\left(u^2\right)'' = 0 \Longrightarrow u(x)^2 = C_1x + C_2$$

   (b) Find u_0, the extremal satisfying the boundary conditions $u_0(a) = 1$ and $u_0(b) = 0$, where $a < b$. Sketch a graph of the extremal.

   Solution With $u_0(a) = 1, u_0(b) = 0$

$$1^2 = C_1a + C_2, 0^2 = C_1b + C_2$$

$$\Longrightarrow C_1 = \frac{1}{a-b}, C_2 = \frac{-b}{a-b}$$

$$u_0(x)^2 = \frac{x-b}{a-b}$$

   (c) Compute $\mathcal{F}(u_0)$ and $\mathcal{F}(u)$ where $u(x)$ is the linear function satisfying $u(a) = 1$ and $u(b) - 0$.

   Solution

$$u_0(x)^2 = \frac{x-b}{a-b}$$

$$2u_0u_0' = \frac{1}{a-b} \Longrightarrow (u_0')^2 = \frac{1}{(2u_0(a-b))^2}$$

Thus

$$\mathcal{F}(u_0) = \int_a^b u_0^2(u_0')^2 dx \tag{1}$$
$$= \int_a^b \frac{u_0^2}{4u_0^2(a-b)^2} dx \tag{2}$$
$$= \int_a^b \frac{1}{4(a-b)^2} dx \tag{3}$$
$$= \frac{1}{4(a-b)^2} x|_a^b \tag{4}$$
$$= -\frac{1}{4(a-b)} \tag{5}$$

For the linear function, $u = \frac{x-b}{a-b}$

$$\mathcal{F}(u) = \int_a^b u^2(u')^2 dx \tag{6}$$
$$= \int_a^b \frac{(x-b)^2}{(a-b)^2} \frac{1}{(a-b)^2} dx \tag{7}$$
$$= \frac{1}{(a-b)^4} \int_a^b (x-b)^2 dx \tag{8}$$
$$= \frac{1}{(a-b)^4} \frac{(x-b)^3}{3} |_a^b \tag{9}$$
$$= -\frac{1}{3(a-b)} \tag{10}$$

2.

$$\mathcal{F}(u) = \int_a^b F(u(x), u'(x)) dx$$

(a) $y = u(x)$, let $v = u^{-1}$, $x = v(y)$. By the inverse function theorem,

$$v'(y) = (u^{-1})'(y) = \frac{1}{(u'(x))}$$

$dx = v'(y)dy$. Plugging this into the functional,

$$\mathcal{F}(u) = \int_a^b F(u(x), u'(x)) dx \tag{11}$$
$$= \int_{u(a)}^{u(b)} F(y, \frac{1}{v'(y)}) v' dy \tag{12}$$
$$= \int_{a'}^{b'} H(y, v, v') dy \tag{13}$$

where $u(a) = a', u(b) = b'$ and $H(y, v, q) = qF(y, 1/q)$.

(b) $H_v = 0$. Therefore the Euler equation for this functional is

$$-\frac{d}{dy} H_q = 0$$

Thus, $H_q = C$, for some constant.

$$\begin{aligned} H_q &= F(y, 1/q) + qF_q(y, 1/q)(-1/q^2) \quad (14) \\ &= F(u, p) - pF_p(y, 1/q) \quad (15) \end{aligned}$$

Thus, $pF_p - F = C$.

3. Use the spherical coordinates

$$x = R\cos(\theta)\sin(\phi), \;\; y = R\sin(\theta)\sin(\phi), \;\; z = R\cos(\phi).$$

(a) Show that the element of arc length on the sphere is given by $ds^2 = R^2 \left[d\phi^2 + \sin^2 \phi d\theta^2\right]$.

Solution Calculate

$$\begin{aligned} dx &= R\left(\cos\theta\cos\phi\, d\phi - \sin\theta\sin\phi\, d\theta\right) \\ dy &= R\left(\sin\theta\cos\phi\, d\phi + \cos\theta\sin\phi\, d\theta\right) \\ dz &= -R\sin\phi\, d\phi \end{aligned}$$

Then

$$\begin{aligned} dx^2 &= R^2\left(\cos^2\theta\cos^2\phi\, d\phi^2 - 2\sin\theta\sin\phi\cos\theta\cos\phi\, d\theta\, d\phi + \sin^2\theta\sin^2\phi\, d\theta^2\right) \\ dy^2 &= R^2\left(\sin^2\theta\cos^2\phi\, d\phi^2 + 2\sin\theta\sin\phi\cos\theta\cos\phi\, d\theta\, d\phi + \cos^2\theta\sin^2\phi\, d\theta^2\right) \\ dz^2 &= R^2\sin^2\phi\, d\phi^2. \end{aligned}$$

Summing, find that

$$\begin{aligned} ds^2 &= R^2\left(\left(\cos^2\theta\cos^2\phi + \sin^2\theta\cos^2\phi + \sin^2\phi\right)d\phi^2 + \left(\sin^2\theta\sin^2\phi + \cos^2\theta\sin^2\phi\right)d\theta^2\right) \\ &= R^2\left(\left(\cos^2\phi + \sin^2\phi\right)d\phi^2 + \sin^2\phi\, d\theta^2\right) \\ &= R^2\left[d\phi^2 + \sin^2\phi\, d\theta^2\right]. \end{aligned}$$

(b) If we limit ourselves to curves which can be parametrized by the polar angle ϕ, with $\theta = w(\phi), a \le \phi \le b$, show that the length of the curve γ is given by

$$\mathcal{F}(w) = \int_\gamma ds = \int_a^b F(\phi, w, w')\, d\phi,$$

where $F(\phi, w, p) = R\sqrt{1 + p^2\sin^2\phi}$.

Solution We are given that $\theta = w(\phi)$, so $d\theta = w'(\phi)d\phi$. With $p = w'(\phi)$,

$$ds^2 = R^2 \left[1 + p^2 \sin^2 \phi\right] d\phi^2.$$

The assertion follows immediately.

(c) Integrate the Euler equation for this functional and use the parametric equations for the sphere to show that there are constants A, B, C such that the extremal curve lies in the plane through the origin defined by $Ax + By + Cz = 0$. Hence you will have proved that the geodesics on the sphere are great circles.

Solution The Euler equation yields $\frac{Rp \sin^2 \phi}{\sqrt{1+p^2 \sin^2 \phi}} = C_1$, or

$$\frac{dw}{d\phi} = \frac{1}{\sin \phi \sqrt{\frac{R^2}{C_1^2} \sin^2 \phi - 1}}.$$

Calculating the integral in Mathematica,

$$w = -\arctan\left(\frac{\cos \phi}{\sqrt{\frac{R^2}{C_1^2} \sin^2 \phi - 1}}\right) + C_2,$$

which using trig manipulations becomes

$$\sin(C_2 - w) = \frac{\cot \phi}{\sqrt{\frac{R^2}{C_1^2} - 1}}.$$

By more trig arrangements,

$$\sin(C_2) \cos w \sin \phi - \cos(C_2) \sin w \sin \phi - \frac{\cos \phi}{\sqrt{\frac{R^2}{C_1^2} - 1}} = 0.$$

Substituting $w = \theta$ and using the spherical parametrizations, we see that $Ax+By+Cz = 0$ is satisfied for $A = \frac{\sin(C_2)}{R}, B = \frac{-\cos(C_2)}{R}, C = \frac{-1}{R\sqrt{\frac{R^2}{C_1^2}-1}}$. Thus the geodesics on the sphere are great circles.

4. Omitted.

5. Considering "signed" area, the functional is $F = \int_a^b (u(x) - \gamma(x))\, dx$ where $\gamma(x)$ is the curve given. We also have the conditions $u(a) = a', u(b) = b', G = \int_a^b \sqrt{1 + u'^2} dx = l$.

   Using Lagrange multipliers to find the extremum of the functional F subject to the constraint G, the Euler equation is

$$F_u - \frac{d}{dx} F_{u'} + \lambda(G_u - \frac{d}{dx} G_{u'}) = 0$$

$$1 - \lambda \frac{d}{dx} \frac{u'}{\sqrt{1+u'^2}} = 0$$
$$\implies \frac{1}{\lambda}(x+C) = \frac{u'}{\sqrt{1+u'^2}}$$

Solving for u', we get

$$u' = \frac{\frac{x+C}{\lambda}}{\sqrt{1-\left(\frac{x+C}{\lambda}\right)^2}}$$

Integrating

$$u = \lambda \sqrt{1 - \left(\frac{x+C}{\lambda}\right)^2} + C_1$$

Rearranging

$$(u - C_1)^2 + (x+C)^2 = \lambda^2.$$

That is, the extremum u is the arc of the circle connecting a and b of length l.

6. Considering the Euler equation for extrema under the constraint,

$$F_u - \frac{d}{dx}F_{u'} + \lambda(G_u - \frac{d}{dx}G_{u'}) = 0$$

If $\lambda = 0$, we are left with

$$F_u - \frac{d}{dx}F_{u'} = 0,$$

which is the Euler equation for finding extremals of $\mathcal{F}$ alone.

7. If u is an extremal for $\mathcal{F}$ subject to constraint $\mathcal{G} = c_1$, then there exists a λ such that

$$F_u - \frac{d}{dx}F_{u'} + \lambda(G_u - \frac{d}{dx}G_{u'}) = 0$$

As long as $\lambda \neq 0$, we can rearrange this equation to give

$$G_u - \frac{d}{dx}G_{u'} + \frac{1}{\lambda}(F_u - \frac{d}{dx}F_{u'}) = 0$$

Therefore, u is also an extremal to $\mathcal{G}$ subject to some constraint $\mathcal{F} = c_2$, as long as $\lambda \neq 0$.