Math 410 Homework Solutions

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1 Homework #4

1. Let S be a surface of revolution about the z-axis. It is defined by an equation of the form $x^2 + y^2 = f(z)$. Let γ be a geodesic on S and let $u(t) = (x(t), y(t), z(t))$ be its parametrization by arc length.

   (a) Derive the Euler-Lagrange equations for x, y, and z.
   Solution Consider the following functional

   $$\mathcal{F} = \int |u'(t)| + \lambda(t)(x^2 + y^2 - f(z))dt$$

   For γ to be a geodesic, u must satisfy the Euler equations for this functional, i.e.

   $$2\lambda x - \frac{d}{dt}\frac{x'}{|u'|} = 0$$
   $$2\lambda y - \frac{d}{dt}\frac{y'}{|u'|} = 0$$
   $$-\lambda\frac{df}{dz} - \frac{d}{dt}\frac{z'}{|u'|} = 0$$

   Using the fact that u is parameterized by arc length, we get

   $$2\lambda x = x''$$
   $$2\lambda y = y''$$
   $$-\lambda\frac{df}{dz} = z''$$

   (b) Show that $xy' - yx' = c_1$, where c_1 is a constant.
   Solution Multiplying the first equation by y, second by x, we find $yx'' = xy''$. But notice

   $$\frac{d}{dt}(xy' - yx') = xy'' + x'y' - y'x' - yx'' = xy'' - yx''$$

   Therefore, $\frac{d}{dt}(xy' - yx') = 0$, i.e. $xy' - yx' = C_1$.

(c) Show that the element of arc length in $\mathbb{R}^3$ is given by $ds^2 = dr^2 + r^2 d\theta^2 + dz^2$.

Solution First calculate that

$$dx = \cos\theta dr - r\sin\theta d\theta$$

$$dy = \sin\theta dr + r\cos\theta d\theta.$$

Then

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + dz^2 \\ &= \cos^2\theta dr^2 + r^2\sin^2\theta d\theta + \sin^2\theta dr^2 + r^2\cos^2\theta d\theta \\ &= dr^2 + r^2 d\theta^2 + dz^2. \end{aligned}$$

(d) Express the parametrization for the geodesic γ in cylindrical coordinates $u(t) = (r(t), \theta(t), z(t))$, and find the Euler-Lagrange equations for r, θ, and z.

Solution Let

$$\mathcal{G} = \int_a^b \sqrt{dr^2 + r^2 d\theta^2 + dz^2} + \lambda(t)(r^2 - f(z))dt.$$

Utilizing the fact that $\sqrt{dr^2 + r^2 d\theta^2 + dz^2} = 1$, the Euler-Lagrange equations are

$$\begin{aligned} r(\theta')^2 + 2r\lambda - r'' &= 0 \\ r^2\theta'' + 2rr'\theta' &= 0 \\ \lambda f'(z) + z'' &= 0. \end{aligned}$$

(e) Show that $r^2\theta' = c_2$. Then show that $c_2 = c_1$ where c_1 is the constant in part (b).

Solution Noting that $\frac{d}{dt}(r^2\theta') = r^2\theta'' + 2rr'\theta' = 0$ by part (d), it is clear that $r^2\theta' = c_2$ for some constant c_2.

From the cylindrical coordinates, we know that $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Thus

$$\begin{aligned} c_1 &= xy' - yx' \\ &= r\cos\theta(r'\sin\theta + r\theta'\cos\theta) - r\sin\theta(r'\cos\theta - r\theta'\sin\theta) \\ &= r^2\theta'\cos^2\theta + r^2\theta'\sin^2\theta \\ &= r^2\theta' \\ &= c_2. \end{aligned}$$

2. Consider the cylinder S in $\mathbb{R}^3$ defined by the equation $x^2 + y^2 = a^2$.

(a) The points $A = (a, 0, 0)$ and $B = (a\cos\alpha, a\sin\alpha, b)$ both lie on S. Find the geodesics joining them.

Solution We can parameterize the cylinder by

$$\mathbf{r}(\theta, z) = (a\cos\theta, a\sin\theta, z).$$

Then for a curve on the surface, we define the arclength functional

$$L(\theta, z) = \int_a^b \sqrt{E\theta'^2 + 2F\theta' z' + Gz'^2}dt$$

where E, F, G are coefficients of the first fundamental form of the surface, i.e.

$$E = \mathbf{r}_\theta \cdot \mathbf{r}_\theta = a^2$$
$$F = \mathbf{r}_\theta \cdot \mathbf{r}_z = 0$$
$$G = \mathbf{r}_z \cdot \mathbf{r}_z = 1$$

Thus, the Euler equations for our functional are:

$$\frac{d}{dt}\frac{a^2\theta'}{\sqrt{a^2\theta'^2 + z'^2}} = 0$$
$$\frac{d}{dt}\frac{z'}{\sqrt{a^2\theta'^2 + z'^2}} = 0$$

$$\Longrightarrow \frac{a^2\theta'}{\sqrt{a^2\theta'^2 + z'^2}} = C_1$$
$$\frac{z'}{\sqrt{a^2\theta'^2 + z'^2}} = C_2$$

Combining these two expressions,

$$z = C_3\theta + C_4$$

However, we need to use boundary conditions, that the geodesic goes through $A = (a, 0, 0), B = (a\cos\alpha, a\sin\alpha, b)$. When $\theta = 0$, $z = C_4 = 0$. Thus, $z = C_3\theta$.

Also, when $\theta = \alpha, z = C_3\alpha = b \Longrightarrow C_3 = b/\alpha$.

$$z = \frac{b}{\alpha}\theta$$

(b) Find two different extremals of the length functional joining $A = (a, 0, 0)$ and $C = (a, 0, 2\pi)$. How many extremals join A and C?

Solution By part (a), it is clear that cylindrical curves $(a, 2k\pi t, t)$ for $0 \leq t \leq 2\pi$ are extremals for any $k \in \mathbb{Z}$, so there are infinitely many extremals.

3. Consider a surface S in $\mathbb{R}^3$ defined by $g(x, y, z) = 0$. A thread has its end fixed at the points P_1 and P_2 on S, and its position is given by the parametrization

$$u(t) = (x(t), y(t), z(t)) \quad 0 \leq t \leq L.$$

Assume that this is the parametrization by arc length. The thread has assumed an equilibrium position under the force of gravity (which points in the negative z direction). This means that the potential energy is minimized, which is equivalent to $\int_0^L z(t)\,dt$ being minimized. Describe how you would proceed to find u. In particular, show the differential equations that u must satisfy. It is not necessary to solve these equations.

Solution This is an extremal problem with two constraints, $g = 0$ and $\int_0^L \sqrt{x'^2 + y'^2 + z'^2} dt = L$. Using Lagrange multipliers, we will find extremum for the new functional:

$$\mathcal{F} = \int_0^L z(t) + \lambda_1 g + \lambda_2 \sqrt{x'^2 + y'^2 + z'^2}$$

The Euler equations for this functional are:

$$\lambda_1 g_x - \frac{d}{dt} \frac{\lambda_2 x'}{\sqrt{x'^2 + y'^2 + z'^2}} = 0$$

$$\lambda_1 g_y - \frac{d}{dt} \frac{\lambda_2 y'}{\sqrt{x'^2 + y'^2 + z'^2}} = 0$$

$$1 + \lambda_1 g_z - \frac{d}{dt} \frac{\lambda_2 z'}{\sqrt{x'^2 + y'^2 + z'^2}} = 0.$$

Bust $\sqrt{x'^2 + y'^2 + z'^2} = 1$, so this reduces to

$$\lambda_1 g_x = \lambda_2 x''$$

$$\lambda_1 g_y = \lambda_2 y''$$

$$1 + \lambda_1 g_z = \lambda_2 z''.$$

4. Suppose that $a < b$. Consider functions $u(x)$ defined on $[a, b]$ which satisfy $u(a) = u(b) = 0$, and $u(x) > 0$ for $a < x < b$. If we rotate the graph of u about the x-axis, we get a surface of revolution that is the boundary of a solid of revolution. Among all of those functions for which the area of the surface of revolution is equal to a fixed number A, we want to find the one for which the volume of the solid is maximum.

   (a) Show that there are constants λ and C for which

$$(u(x))^2 + \frac{\lambda u(x)}{\sqrt{1 + (u'(x))^2}} = C.$$

   Solution We will find extremals for the functional

$$\mathcal{F}(u) = \int_a^b \left(\pi u^2 + 2\lambda \pi u \sqrt{1 + (u')^2} \right) dx.$$

   Because there is no dependence on x, the Euler-Lagrance equation becomes

$$C_1 = u' \frac{2\pi\lambda u u'}{\sqrt{1 + (u')^2}} - \pi u^2 - 2\pi\lambda u \sqrt{1 + (u')^2}$$
$$C_2 = u' \frac{2\lambda u u'}{\sqrt{1 + (u')^2}} - u^2 - \frac{2\lambda u (1 + (u')^2)}{\sqrt{1 + (u')^2}}$$
$$C_2 = -u^2 - \frac{2\lambda u}{\sqrt{1 + (u')^2}},$$

   which is in the proper form, renaming λ for 2λ, and letting $C = -C_2 = \frac{C_1}{\pi}$.

   (b) Use the boundary conditions on u to show that $C = 0$.
   Solution Because $u(a) = 0$, it is clear that $C = 0^2 + \frac{0}{\sqrt{1+(u'(a))^2}} = 0$.

   (c) Find u explicitly, and describe the solid.
   Solution Because $u(x) \neq 0$ for $a < x < b$, the equation from part (a) reduces to

$$u + \frac{\lambda}{\sqrt{1 + (u')^2}} = 0.$$

   The solution to this differential equation is

$$u(x) = \sqrt{\lambda^2 - (x - c)^2}$$

   for some constant c. Thus u is an arc of a circle going through the points $(a, 0)$ and $(b, 0)$ with center $(c, 0)$. Because this circle is centered on the line between $(a, 0)$ and $(b, 0)$, the center has to be $(\frac{b+a}{2}, 0)$, making the radius $b - \frac{b+a}{2} = \frac{b-a}{2}$. Thus the extremal is

$$u(x) = \sqrt{\left(\frac{b-a}{2}\right)^2 - \left(x - \frac{b+a}{2}\right)^2}.$$

The solid of revolution is simply a ball of radius $\frac{b-a}{2}$ centered at $(\frac{b+a}{2}, 0, 0)$ in $\mathbb{R}^3$.