1 Homework #5

1. Find the extremals of

$$\mathcal{F}(u) = \int_0^1 \left(a + u''(x)^2\right) \, dx$$

with end conditions $u(0) = 0, u'(0) = 1, u(1) = 1$, and $u'(1) = 1$.

Solution Using the formula on page 42, the Euler-Lagrange equation is

$$F_y - \frac{d}{dx}F_{y'} + \frac{d^2}{dx^2}F_{y''} = 0,$$

which yields

$$0 = 0 - 0 + \frac{d^2}{dx^2}\left(2u''(x)\right) = u^{(4)}(x).$$

Integrating yields $u(x) = c_3x^3 + c_2x^2 + c_1x + c_0$ for some constants. Evaluating for the given boundary conditions gives $u(x) = x$.

2. Find the shortest curve(s) $y = u(x)$ for $a \leq x \leq b$ with the endpoints $u(a) = c$ and $u(b) = d$ unspecified. Solve the problem without doing any computations.

Solution Let γ_a and γ_b be curves in $\mathbb{R}^2$ defined by $\gamma_a(t) = (a, t)$ and $\gamma_b(t) = (b, t)$ for all $t \in \mathbb{R}$. Then the problem is to find an extremal curve of $\int_{x_1}^{x_2} f(x, y)\sqrt{1 + y'^2}\, dx$ going from the image of γ_a to the image of γ_b, where $f(x, y) \equiv 1$. We know by solving the Euler-Lagrange equation that extremals will be straight lines. Also, for functionals of the above form, the tranvsersality conditions reduce to orthogonality. That is, extremal curves need to be perpendicular to both γ_a and γ_b. Thus the shortest curves that we seek are horizontal lines from γ_a to γ_b.

3. Find the shortest curve(s) in the plane connecting points on two disjoint circles. Solve the problem without doing any computations.

Solution By the above problem, the shortest curve must be a straight line which is perpendicular to both circles. To achieve this latter condition, the extremal curve must lie on the line segment connecting the centers of the two circles, but consists of only those points between the two circles.

4. Find the distance in the plane between the line $y = x$ and the parabola $y = \frac{9}{4} + x^2$.

Solution As in the previous two problems, the shortest curve γ between the straight line and the parabola is a straight line perpendicular to both. Thus, γ intersects the parabola at a point where the tangent line of the parabola is parallel to the line $y = x$. Solving

$$\frac{d}{dx}\left(\frac{9}{4} + x^2\right) = 2x = 1,$$

we find that γ intersects the parabola at the point $(\frac{1}{2}, \frac{5}{2})$. We also note that γ has slope -1. Using point-slope form, γ is characterized by

$$y - \frac{5}{2} = -\left(x - \frac{1}{2}\right),$$

or that $y = -x + 3$.

We find that γ intersects the line $y = x$ at the point $(\frac{3}{2}, \frac{3}{2})$. The distance between the two points $(\frac{3}{2}, \frac{3}{2})$ and $(\frac{1}{2}, \frac{5}{2})$, which is $\sqrt{2}$, is the distance between the two given curves.

5. Find the Euler equation that must be satisfied by an extremal of the functional

$$\mathcal{F}(u) = \int_a^b F(x, u(x), u'(x), u''(x), u'''(x))\, dx.$$

Solution Let $v = u', w = u''$. Then the original problem reduces to finding the extremum of

$$\mathcal{G}(u, v, w) = \int_a^b F(x, u, u', v', w')dx$$

subject to the constraints $v = u', w = v'$.

Let

$$H(x, u, u', v, v', w, w') = F(x, u, u', v', w') + \lambda_1(v - u') + \lambda_2(w - v')$$

Then the Euler equations for this functional are:

$$F_u - \frac{d}{dx}(F_{u'} - \lambda_1) = 0 \quad (1)$$

$$\lambda_1 - \frac{d}{dx}(F_{v'} - \lambda_2) = 0 \quad (2)$$

$$\lambda_2 - \frac{d}{dx}F_{w'} = 0 \quad (3)$$

Plugging (3) into (2), and then (2) into (1), we get

$$\frac{d^3}{dx^3}F_{u'''} - \frac{d^2}{dx^2}F_{u''} + \frac{d}{dx}F_{u'} - F_u = 0$$

6. Find the extremals of the functional

$$\mathcal{F}(u) = \int_0^{\pi/4} (u^2 - u'^2)\,dx$$

with $u(0) = 0$ and $u\left(\frac{\pi}{4}\right)$ unrestricted.

Solution The Euler equation is:

$$F_u - \frac{d}{dx}F_{u'} = 0 \Longrightarrow 2u + 2u'' = 0 \Longrightarrow u'' = -u$$

Solving this equation, $u(x) = C_1\cos(x) + C_2\sin(x)$, constants C_1, C_2. Since the one boundary condition is given, $u(0) = 0 \Longrightarrow C_1 = 0$. Using the natural boundary condition$F_{u'}|_{\pi/4} = 0$, $-2u'(\pi/4) = 0 \Longrightarrow C_2 = 0$. Hence, the extremal is $u(x) = 0$.

7. Find the extremals of the functional

$$\mathcal{F}(u) = \int_0^b \frac{\sqrt{1+u'^2}}{u}\,dx$$

with $u(0) = 0$ and $g(b, u(b)) = 0$, where

(a) $g(x, y) = x - y - 5$

Solution Since F does not depend on x, we use $F - pF_p = C$ to solve the Euler equations:

$$\frac{\sqrt{1+p^2}}{u} - \frac{p^2}{u\sqrt{1+p^2}} = C$$

$$\frac{1}{u\sqrt{1+p^2}} = C \Longrightarrow u^2(1+p^2) = C_1^2$$

Solving this equation, $u(x) = \pm\sqrt{C_1^2 - (x + C_2)^2}$.
Given that $u(0) = 0$, it follows that $C_1^2 = C_2^2$.
For the other boundary condition,

$$F_p(b, u(b), u'(b) = F(b, u(b), u'(b))\frac{g_y}{g_x + g_y u'}$$

$$\frac{u'(b)}{u(b)\sqrt{1+u'(b)^2}} = \frac{\sqrt{1+u'(b)^2}}{u(b)}\frac{-1}{1-u'(b)}$$

$$\Longrightarrow u'(b) = -1$$

$$u'(b) = \mp\frac{b + C_1}{\sqrt{-b^2 - 2C_1 b}} = -1$$

$$g(b,u(b)) = 0 \Longrightarrow b-5 = u(b) = \pm\sqrt{-b^2-2C_1 b}$$

Using these two equations, we find

$$\frac{b+C_1}{b-5} = 1 \Longrightarrow C_1 = -5$$

Thus the extremal is $u(x) = \pm\sqrt{-x^2+10x}$

(b) $g(x,y) = (x-9)^2 + y^2 - 9$;

Now for the other boundary condition, we must satisfy

$$F_p(b,u(b),u'(b) = F(b,u(b),u'(b))\frac{g_y}{g_x + g_y u'}$$

$$\frac{u'(b)}{u(b)\sqrt{1+u'(b)^2}} = \frac{\sqrt{1+u'(b)^2}}{u(b)}\frac{u(b)}{b-9+u(b)u'(b)}$$

$$\Longrightarrow \frac{u'(b)}{1+u'(b)^2} = \frac{u}{(b-9)+u(b)u'(b)}$$

Cross multiplying and reducing, we get

$$u'(b)(b-9) = u(b) \Longrightarrow b = \frac{-9C_1}{9+C_1}$$

Also,

$$g(b,u(b)) = 0 \Longrightarrow (b-9)^2 + u(b)^2 - 9 = 0$$

.

Plugging in our b (in terms of C_1) into g, we get:

$$\left(\frac{9C_1}{9+C_1}+9\right)^2 + \left(-(\frac{9C_1}{9+C_1})^2 + 2(\frac{9C_1^2}{9+C_1})\right) = 9$$

Solving for C_1, we get $C_1 = -4$, thus

$$u(x) = \pm\sqrt{-x^2+8x}$$

8. Describe the curve connecting two circles in the plane along which a particle will fall in minimum time under the force of gravity. In addition, write down the equations which must be solved to find the constants involved.

Solution The curve must satisfy the Euler-Lagrange equation from the brachistochrone problem. That is, the curve must be given by $(x(t),y(t))$, where

$$x(t) = a + k\,(t - \sin t)\,, \text{ and}$$
$$y(t) = c - k\,(1-\cos t)$$

for some constants a, c, and k.

The transversality conditions require that the curve (x, y) also be perpendicular to the given circles. Let

$$c_1(\tau) = (x_1 + r_1 \cos\tau, y_1 + r_1 \sin\tau) \text{ and}$$
$$c_2(\tau) = (x_2 + r_2 \cos\tau, y_2 + r_2 \sin\tau)$$

parametrize the circles. Thus we must solve

$$(x'(t_1), y'(t_1)) \cdot c_1'(\tau_1) = 0 \text{ and}$$
$$(x'(t_2), y'(t_2)) \cdot c_1'(\tau_2) = 0$$

to find the necessary constants.