1 Homework #6

1. Hilbert's theorem says: Suppose that $F(x,u,p)$ is continuously differentiable for $a \le x \le b$ and for all $(u,p) \in \mathbb{R}^2$. Suppose that F_p is continuously differentiable and that $F_{pp}(x,u,p) > 0$ for $a \le x \le b$ and for all $(u,p) \in \mathbb{R}^2$. If u is a weak piecewise continuously differentiable local extremal for $\mathcal{F}(u) = \int_a^b F(x,u(x),u'(x))\,dx$, then $u \in \mathcal{C}^2[a,b]$.

   (a) Show that the piecewise continuously differentiable function

   $$u(x) = \begin{cases} x & \text{for } 0 \le x \le 1/2 \\ 1-x & \text{for } 1/2 \le x \le 1 \end{cases}$$

   is a global minimum for the functional

   $$\mathcal{F}(u) = \int_0^1 \left(u'(x)^2 - 1\right)^2\,dx.$$

   Solution Note that $F(x,u,p) = (p^2-1)^2$ is non-negative, so if there is a function u satisfying $F(x,u,u') = 0$ for all but a finite number of x values in $[0,1]$, then such a u is obviously a global minimum. Using the given u, it is clear that $F(x,u,u') = 0$ for all $x \in [0,1]$ except for $x = \frac{1}{2}$, where the derivative of u is not defined. Immediately, we have $\mathcal{F}(u) = 0$ for the given u, so that u must be a global minimum.

   (b) Why is this example not a counterexample to Hilbert's theorem?
   Solution It is quickly calculated that

   $$F_{pp} = 12p^2 - 4,$$

   which may not always be positive, as the theorem would require. Specifically, $F_{pp} < 0$ if $|p| < \frac{1}{\sqrt{3}}$.

2. Prove that the functional

   $$\mathcal{F}(u) = \int_a^b \left(Au'(x)^2 + Bu(x)u'(x) + Cu(x)^2 + Du'(x) + Eu(x)\right)\,dx$$

   has no broken extremals, where A, B, C, D, and E are constants and $A \neq 0$.

   Solution Suppose for contradiction that there does exist a broken extremal u. Let $x_0 \in (a,b)$ be the corner point, that is u' is discontinuous at x_0. With $F(x,u,p)$ defined by the given integrand, it follows that $F_p = 2Ap + Bu + D$. Define

   $$F_p(x_0,u(x_0),p_+) = \lim_{x\to x_0^+} F_p(x,u(x),u'(x)) \text{ and}$$
   $$F_p(x_0,u(x_0),p_-) = \lim_{x\to x_0^-} F_p(x,u(x),u'(x)).$$

The Weierstrass-Erdmann conditions force

$$\begin{aligned}F_p(x_0, u(x_0), p_+)) &= F_p(x_0, u(x_0), p_-)), \text{ so}\\ 2Ap_+ Bu(x_0) + D &= 2Ap_- + Bu(x_0) + D, \text{ then}\\ p_+ &= p_-,\end{aligned}$$

since $A \neq 0$ and u is continuous (at x_0). This means that $\lim_{x \to x_0} u'(x)$ exists, as the right and left limits are equal. This contradicts the broken extremal condition, so $\mathcal{F}$ can have no broken extremals.

Alternately, an argument by Hilbert's theorem is also acceptable.

3. Does the functional

$$J[y] = \int_0^{x_1} (y')^3 \, dx, \quad y(0) = 0, \quad y(x_1) = y_1$$

have broken extremals?

Solution Define F, x_0, p_+, and p_- as in the previous problem. By the corner conditions,

$$\begin{aligned}3p_+^2 &= 3p_-^2 \text{ and}\\ -2p_+^3 &= -2p_-^3.\end{aligned}$$

This only holds when $p_+ = p_-$, so by the previous problem's reasoning, there are no broken extremals of J.

4. Find the extremals of the functional

$$J[y] = \int_0^4 (y' - 1)^2 (y' + 1)^2 \, dx, \quad y(0) = 0, \quad y(4) = 2$$

which have only one corner.

Solution Let $p_+ = \lim_{x \to x_0^+} u'(x)$ and $p_- = \lim_{x \to x_0^-} u'(x)$. Calculate that $F_p = 4(p^3 - p)$. Using the Weierstrass-Erdmann conditions for F_p, find that

$$(p_- - p_+)(p_-^2 + p_- p_+ + p_+^2 - 1) = 0.$$

Ignoring the case when $p_- = p_+$, which would signify no discontinuity of u', find that $(p_-^2 + p_- p_+ + p_+^2 - 1) = 0$.

Using the Weierstrass-Erdmann conditions for $F - pF_p$, find that

$$(p_- - p_+)(p_+ + p_-)(3p_-^2 + 3p_+^2 - 2) = 0.$$

Again ignoring the case when $p_- = p_+$, which would signify no discontinuity of u', find that $\pm p_- = \mp p_+$, or (from combining all other equations) that $p_+ = p_- = \pm\frac{\sqrt{3}}{3}$, which again is ignored.

Since $F_x = 0$, we know that $F - pF_p$ is constant. Solving $F - pF_p$ for $p = 1$ and $p = -1$ shows that this constant must be zero. Solving then for the real roots of $F - pF_p$ yield that $u'(x) = \pm 1$ where the derivative exists.

Some graphical analysis shows that the only curves whose straight line components have slope ± 1 that satisfy the boundary conditions are

$$u(x) = \begin{cases} x & \text{for } 0 \leq x \leq 3 \\ -x + 6 & \text{for } 3 \leq x \leq 4 \end{cases}$$

and

$$u(x) = \begin{cases} -x & \text{for } 0 \leq x \leq 1 \\ x - 2 & \text{for } 1 \leq x \leq 4. \end{cases}$$

5. Same as #3.

6. Consider the functional

$$\mathcal{F}(u) = \int_{-1}^{1} u(x)^2 (1 - u'(x))^2 \, dx. \tag{1}$$

   (a) Find extremals of $\mathcal{F}$ with $u(-1) = 5$ and $u(1) = 3$.
   Solution Computing the Euler-Lagrange equation for $F - pF_p$ constant yields hyberbolic solutions

$$x = \pm\sqrt{u^2 + a} + b.$$

   Noting that $F_p = 2u^2(1 - p)$, we explore the possibility of broken extremals. For F_p to be continuous with p discontinuous, u must be zero. For our given boundary conditions, no such u will occur, so proceed as for smooth extremals.
   Solving for the boundary conditions, find that $a = 0$ and $b = 4$. Taking the choice of negative sign fits the boundary conditions, so therefore $u(x) = 4 - x$ is the extremal.

   (b) Find extremals of $\mathcal{F}$ with $u(-1) = 5$ and $u(1) = 1$.
   Solution Using the same solution and a, b as above, find that $a = 24$ and $b = 6$ satisfy the boundary conditions. Again, choose the negative sign. The extremal is $u(x) = \sqrt{(x-6)^2 - 24}$.

7. Again we consider the variational integral in (1).

   (a) Find a candidate $u \in PC^1[-1, 1]$ for a local minimum of $\mathcal{F}$ with

$$u(-1) = -\frac{1}{2} \text{ and } u(1) = \frac{1}{2}.$$

   Solution Because the integrand is non-negative, a function u that makes $\mathcal{F}(u) = 0$ suffices. It is easily verified that

$$u(x) = \begin{cases} x + \frac{1}{2} & \text{for } -1 \leq x \leq -\frac{1}{2} \\ 0 & \text{for } -\frac{1}{2} \leq x \leq \frac{1}{2} \\ x + \frac{1}{2} & \text{for } \frac{1}{2} \leq x \leq 1 \end{cases}$$

will work.

(b) Find a candidate $u \in PC^1[-1,1]$ for a local minimum of $\mathcal{F}$ with

$$u(-1) = 1 \text{ and } u(1) = 1.$$

Solution With the above reasoning, $u(x) = |x|$ works.