1 Homework #6

1. Consider $f(x) = e^x$ for $x \in \mathbb{R}$.

   (a) Show that f is convex and compute the Legendre transform $f^*(\xi)$.
   Solution It is clear that $f''(x) = e^x > 0$, hence f is convex. Set $\xi = f'(x) = e^x$ and solve for x: $\ln \xi = x$. Then $f^*(\xi) = x\xi - f(x) = \xi \ln \xi - f(\ln \xi) = \xi(\ln \xi - 1)$.

   (b) Prove that $e^n x \leq e^x + (n-1)e^n$ for all $x \in \mathbb{R}$, and all positive integers n.
   Solution Consider the function $g(y) = e^y - y$. Then $g'(y) = e^y - 1$ and has only one root when $y = 0$. So $g''(y) = e^y$ and $g''(0) > 0$. Hence, $y = 0$ is global minimum of g, by the second derivative test. Therefore, $g(y) \geq g(0) = 1 \Longrightarrow e^y - y \geq 1$.
   Let $y = x - n$. Then $e^{x-n} - x + n \geq 1$. Since $e^n > 0$, multiply expression by e^n:

   $$e^x - xe^n + ne^n \geq e^n \Longrightarrow xe^n \leq e^x + (n-1)e^n,$$

   as desired.
   Also note that Young's Inequality $x\xi \leq f(x) + f^*(\xi)$ with $\xi = e^n$ proves the desired result as well.

2. Consider $f(x, y) = e^{(x^2+y^2)/2}$ for $(x, y) \in \Omega$, where

   $$\Omega = \left\{(x, y) \in \mathbb{R}^2 \mid |x| < 1 \text{ and } |y| < 1\right\}.$$

   (a) Show that the Hessian matrix

   $$Hf(x, y) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

   is a positive definite matrix for all $(x, y) \in \Omega$.
   Solution One way to show the matrix is positive definite is to show all the principle minors are positive. In this case, show that $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$. Then

   $$\begin{aligned} f_{xx} &= e^{\frac{x^2+y^2}{2}}(x^2+1) > 0 \\ f_{xx}f_{yy} - f_{xy}^2 &= e^{\frac{x^2+y^2}{2}}(x^2+1)e^{\frac{x^2+y^2}{2}}(y^2+1) - e^{\frac{x^2+y^2}{2}}(x^2)e^{\frac{x^2+y^2}{2}}y^2 \\ &= e^{x^2+y^2}(x^2+y^2+1) > 0. \end{aligned}$$

   Therefore, $Hf(x, y)$ is positive definite.

   (b) Let $\phi(x, y) = \nabla f(x, y)$. Compute and sketch $\Omega^* = \phi(\Omega)$. Show that Ω^* is not convex.

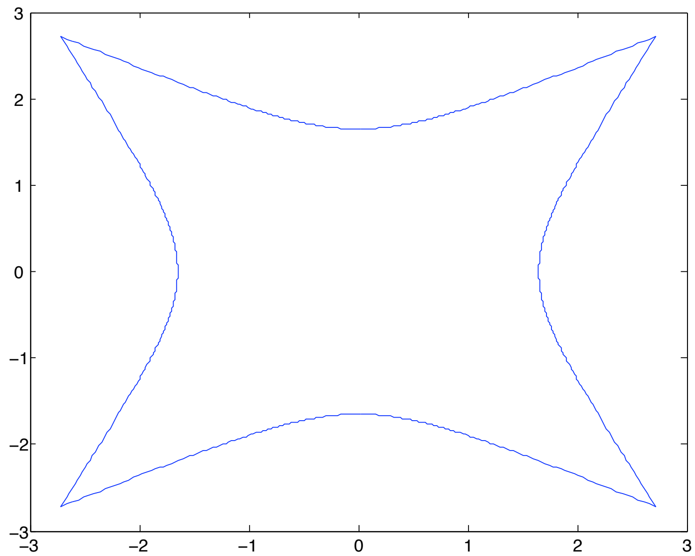

Figure 1: Region Ω^*

Solution Calculating,

$$\phi(x,y) = \nabla f(x,y) = (e^{\frac{x^2+y^2}{2}}x, e^{\frac{x^2+y^2}{2}}y)$$

Observe that the points $(e,e),(e,-e)$ lie on the boundary of this region, and the straight line connecting them is a vertical line. This vertical line intersects the hortizontal axis at e, but the point on boundary of the region on the horizontal axis is $\sqrt{e}$. Thus, the line segment is not contained in the region, so it is not convex.

(c) Show that the function $\alpha(t) = te^t$ is an increasing function on $[0,\infty)$ and maps $[0,\infty)$ onto itself. Let $\beta(s)$ be the inverse of α, so that $s = te^t$ if and only if $t = \beta(s)$. Show that

$$f^*(\xi,\eta) = \left[\beta(\xi^2+\eta^2) - 1\right] e^{\beta(\xi^2+\eta^2)/2}.$$

Solution First note that $\alpha'(t) = e^t + te^t = (t+1)e^t > 0$ when $t > 0$. Thus, $\alpha(t)$ is increasing.

See the $\alpha(0) = 0$ and $\lim_{t\to\infty}\alpha(t) = \infty$, thus α is not bounded above. Altogether, and using that α is continuous and increasing, $\alpha(t)$ maps $[0,\infty)$ onto itself.

Now introduce $\xi = f_x = e^{\frac{x^2+y^2}{2}}x$ and $\eta = f_y = e^{\frac{x^2+y^2}{2}}y$, so that

$$\xi^2 + \eta^2 = e^{x^2+y^2}(x^2+y^2) = \alpha(x^2+y^2).$$

Therefore, $x^2 + y^2 = \beta(\xi^2 + \eta^2)$.
Calculating f^*, find that

$$\begin{aligned} f^*(\xi, \eta) &= (x, y) \cdot (\xi, \eta) - f(x, y) \\ &= (x, y) \cdot (e^{\frac{x^2+y^2}{2}} x, e^{\frac{x^2+y^2}{2}} y) - e^{\frac{x^2+y^2}{2}} \\ &= (x^2 + y^2) e^{\frac{x^2+y^2}{2}} - e^{\frac{x^2+y^2}{2}} \\ &= (\beta(\xi^2 + \eta^2) - 1)(e^{\frac{\beta(\xi^2+\eta^2)}{2}}) \end{aligned}$$

3. Consider the functional with Lagrangian $F(u, p) = \frac{1}{2}(p + ku)^2$, where k is a nonzero constant.

   (a) Find the canonical coordinates and the Hamiltonian.
   Solution The canonical coordinates are $\Pi = F_p = p + ku$ and $p = \Pi - ku$. The Hamiltonian is then

   $$H(x, u, \Pi) = \Pi p - F = \Pi(\Pi - ku) - 1/2(\Pi - ku + ku) = (1/2)\Pi^2 - \Pi ku.$$

   (b) Find the extremals of the functional by solving the Euler equations in their Hamiltonian form.
   Solution Euler equations:

   $$\frac{du}{dx} = \frac{\partial H}{\partial \Pi} = \Pi - ku$$

   $$\frac{d\Pi}{dx} = -\frac{\partial H}{\partial u} = \Pi k$$

   $\Longrightarrow \Pi(x) = C_1 e^{kx}$, plugging this into the first equation for u, we get

   $$u' + ku = C_1 e^{kx}$$

   $$u(x) = \frac{C_1}{2k} e^{kx} + C_2 e^{-kx}$$

   or

   $$u(x) = Ae^{kx} + Be^{-kx}$$

4. Consider the functional with Lagrangian $F(u, p) = p^2/2 - up$.

   (a) Find the canonical coordinates and the Hamiltonian.
   Solution The canonical coordinates are $\Pi = F_p = p - u$ and $p = \Pi + u$. The Hamiltonian is then

   $$H(x, u, \Pi) = \frac{1}{2}\Pi^2 + u\Pi + \frac{1}{2}u^2.$$

(b) Find the extremals of the functional by solving the Euler equations in their Hamiltonian form.

Solution Euler equations:

$$\frac{du}{dx} = \frac{\partial H}{\partial \Pi} = \Pi + u$$
$$\frac{d\Pi}{dx} = -\frac{\partial H}{\partial u} = -\Pi - u$$

$\Longrightarrow C = u + \Pi = u + p - u = u'$, so that

$$u(x) = C_1 x + C_2.$$

5. A vibrating spring with mass m moves in a straight line with spring equilibrium at $x = 0$, under the influence of of a restoring force $F = -kx$.

(a) Show that the Lagrangian is $L(x, p) = \frac{1}{2}mp^2 - \frac{1}{2}kx^2$.

Solution With $u(x)$ the potential energy, $-\frac{d}{dx}u = F = -kx$. Thus, $u(x) = (1/2)kx^2 + C$. Take $C = 0$. Then the Lagrangian is $L(x, p) = T - u = (1/2)mp^2 - (1/2)kx^2$.

(b) Find the canonical coordinates and the Hamiltoninan, and provide a physical interpretation.

Solution The coordinates are

$$\Pi = \frac{\partial L}{\partial p} = mp, p = \Pi/m.$$

The Hamiltonian is

$$H(t, x, \Pi) = \Pi p - L = \Pi^2/m - (1/2)m(\Pi/m)^2 + (1/2)kx^2 = (1/2)\Pi^2/m + (1/2)kx^2.$$

Since $\Pi = mp$ and p is the velocity, we find that Π is simply the momentum. Since $\frac{\Pi^2}{2m} = T$, it is clear that H is the total energy.

(c) Find the extremals.

Solution

$$\frac{\partial H}{\partial \Pi} = \frac{dx}{dt} = \Pi/m$$
$$\frac{\partial H}{\partial x} = -\frac{d\Pi}{dt} = kx$$

$$\frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d\Pi}{dt}\frac{1}{m} \Longrightarrow x'' = \frac{-kx}{m}$$

Therefore, since $k > 0$, the extremals are

$$x(t) = A \sin\left(\sqrt{\frac{k}{m}}t\right) + B\cos\left(\sqrt{\frac{k}{m}}t\right)$$